Functions of Substitution Tilings as a Jacobian

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Abstract

A tiling τ of the Euclidean space gives rise to a function f_τ, which is constant $1/\left|T\right|$ on the interior of every tile T. In this paper we give a local condition to know when f_τ, that is defined by a primitive substitution tiling of the Euclidean space, can be realized as a Jacobian of a biLipschitz homeomorphism of $\mathbb{R}^d$. As an example we show that this condition holds for any star-shaped substitution tiling of $\mathbb{R}^2$. In particular the result holds for any Penrose tiling.

1 Introduction

Definition 1.1. Let τ be a tiling of $\mathbb{R}^d$. Define a function $f_\tau : \mathbb{R}^d \to \mathbb{R}$ by

$$f_\tau(x) = \begin{cases} \frac{1}{\left|T\right|}, & \text{there exists a tile } T \text{ in } \tau \text{ such that } x \in int(T) \\ 0, & \text{otherwise} \end{cases}$$

where $\left|T\right|$ is the volume of T, and $int(T)$ is the interior of T.

We study the question whether the function f_τ, of a given tiling of $\mathbb{R}^d$, can be realized as the Jacobian of a biLipschitz homeomorphism of the space. This question is related to the following question about separated nets: Given a separated net $Y \subseteq \mathbb{R}^d$, is it biLipschitz equivalent to $\mathbb{Z}^d$? i.e., is there a biLipschitz bijection $\phi : Y \to \mathbb{Z}^d$? The connection between these questions can be divided into two parts. First, it was shown by Burago and Kleiner in [BK98] and independently by McMullen in [McM98], that every separated net is biLipschitz to $\mathbb{Z}^d$ if and only if every function $f \in L^\infty(\mathbb{R}^d)$, with $\inf f > 0$, can be realized as a Jacobian of a biLipschitz homeomorphism of $\mathbb{R}^d$. From this they deduced that there is a separated net which is not biLipschitz equivalent to $\mathbb{Z}^d$. Secondly, if τ is a tiling of $\mathbb{R}^d$, and Y is a separated net which is obtained from τ by placing one point in each tile, then knowing that f_τ can be realized as a Jacobian implies that Y is biLipschitz to $\mathbb{Z}^d$ (see [BK02] Lemma 4.1, and [McM98] Theorem 4.1). It was shown in [S11] that primitive substitution tilings give rise to separated nets which are biLipshitz equivalent to $\mathbb{Z}^2$. In

view of the discussion above, McMullen raised the question whether the functions f_τ are Jacobians. This question is answered affirmatively in this paper.

In this paper we will not consider general functions f but rather only functions which, via Definition 1.1, come from tilings of $\mathbb{R}^d$ with finitely many distinct tiles, up to isometry. To motivate this, recall that every tiling τ of $\mathbb{R}^d$ gives rise to a separated net Y_τ, simply by placing a point in each tile. It is easy to verify that any two such nets are in the same biLipschitz equivalence class. We claim that the converse is also true, namely, for every separated net Y one can define a tiling τ_Y such that any Y_{τ_Y} is biLipschitz equivalent to Y. The tiling τ_Y can be defined in a similar way to the Voronoi cells. Divide the plane to small enough dyadic cubes Q, with respect to the constant that determines the minimal distance between two points in Y. To every $y \in Y$ assign the tile

$$T_y = \bigcup \{\text{closed cubes } Q : Q \text{ is closer to } y \text{ than to any other } z \in Y\} .$$

Squares with equal minimal distance to several points, can be added arbitrarily to a T_y, for one of the y's with minimal distance from it. It is easy to see that the tiling τ_Y, which is defined by these tiles, satisfies the requirements, and also has finitely many different tiles. By the result from [BK02] and [McM98] that we mentioned above, if Y is a separated net which is not biLipschitz to $\mathbb{Z}^d$, then f_{τ_Y} cannot be realized as a Jacobian. Moreover, it follows from this construction that when studying which nets are biLipschitz equivalent to $\mathbb{Z}^d$, it is enough to consider separated nets that come from tilings with finitely many tiles.

Since not all f_τ can be realized as a Jacobian, we restrict ourselves to a special class of tilings - substitution tilings. In our main theorem, Theorem 1.2, we show that f_τ of a primitive substitution tiling can be realized as a Jacobian if a local condition on the basic tiles is satisfied.

In [BK02] Burago and Kleiner gave a sufficient condition for a separated net to be biLipschitz equivalent to $\mathbb{Z}^2$. In fact, they gave a condition that says when a function $f : \mathbb{R}^2 \to (0, \infty)$, which is constant on unit lattice squares and with $\inf f > 0$, can be realized as a Jacobian of a biLipschitz homeomorphism of the plane. Their proof uses a sequence of partitions of the plane to larger and larger dyadic squares. The main idea of our proof is to look at this partition to dyadic squares as a special case of a substitution tiling of the space. The second ingredient of our proof is a property of primitive substitution tilings that was obtained in [S11], see Proposition 2.5.

In the main results that are stated below we use basic terminology from the theory of substitution tiling. We refer the reader to §2 for the definitions of those terms. In the context of substitution tiling, every tile T has a natural partition to smaller tiles, induced by the partitions of the substitution rule on the finite collection of basic tiles. We say that a function $f : T \to \mathbb{R}$ is a *weight function* if it is constant and positive on the interiors of the tiles in the partition of T.

Theorem 1.2. *Let τ be a primitive substitution tiling of $\mathbb{R}^d$. Suppose that there is a constant C with the following property: for every basic tile $\mathcal{T}$, and for every weight function $f : \mathcal{T} \to (0, \infty)$, there is a biLipschitz homeomorphism $\varphi : \mathcal{T} \to \mathcal{T}$ with*

$$(1) \qquad \varphi|_{\partial \mathcal{T}} = id, \quad Jac(\varphi) = \frac{|\mathcal{T}|}{\int_\mathcal{T} f} \cdot f \quad a.e. \quad and \quad biLip(\varphi) \leq \left(\frac{\max f}{\min f}\right)^C .$$

Then there exists a biLipschitz homeomorphism $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ *with* $Jac(\Phi) = f_\tau$ *a.e.*

Remark 1.3. *The assumption in the theorem is stronger than what actually needed. In the proof we only use the assumption for weight functions with values which are averages of f_τ on tiles of the τ_m's. In particular, we only use it for bounded functions, $f(x) \in [1/L, L]$, where $L > 0$ depend on τ.*

Theorem 1.4. *For any primitive star-shaped substitution tiling τ of $\mathbb{R}^2$, there is a biLipschitz homeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $Jac(\Phi) = f_\tau$ a.e.*

Corollary 1.5. *For any Penrose tiling τ there is a biLipschitz homeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $Jac(\phi) = f_\tau$ a.e.*

We remark that in the private case where we tile $\mathbb{R}^d$ by lattice cubes (each cube is divided to 2^d cubes), f_τ is a constant function. However, one may ask when a weight function which is constant on lattice cubes is a Jacobian? This question was answered very recently in [ACG11], where they extend the main result of [BK02] to higher dimensions.

2 Preliminaries and Previous Results

A set $T \subseteq \mathbb{R}^d$ is a *tile* if it is homeomorphic to a closed d-dimensional ball. A *tiling* of a set $U \subseteq \mathbb{R}^d$ is a countable collection of tiles, with pairwise disjoint interiors, such that their union is equal to U. A tiling P of a bounded set $U \subset \mathbb{R}^d$ is called a *patch*. We call the set U the *support of P* and we denote it by $supp(P)$. Given a collection of tiles $\mathcal{F}$, denote by $\mathcal{F}^*$ the set of all patches by the elements of $\mathcal{F}$.

Let $\xi > 1$ and let $\mathcal{F} = \{T_1, \ldots, T_k\}$ be a set of d-dimensional tiles.

Definition 2.1. A *substitution* is a mapping $H : \mathcal{F} \to \xi^{-1}\mathcal{F}^*$ such that $supp(T_i) = supp(H(T_i))$ for every i. Namely, it is a set of dissection rules say how to divide the tiles to other tiles from $\mathcal{F}$, with a smaller scale. We may also apply H to collections of tiles. The constant ξ is called the *inflation constant* or the *dilation constant* of H.

Definition 2.2. Let H be a substitution defined on $\mathcal{F}$. Consider the following set of patches:

$$\mathcal{P} = \{(\xi H)^m(T) : m \in \mathbb{N}, T \in \mathcal{F}\}.$$

The *substitution tiling space* X_H is the set of all tilings of $\mathbb{R}^d$ that for every patch P in them there is a patch $P' \in \mathcal{P}$ such that P is a sub-patch of P'. Every tiling $\tau \in X_H$ is called a *substitution tiling* of H.

Consider the following equivalence relation on tiles: $T_i \sim T_j$ if there exists an isometry O such that $T_i = O(T_j)$ and $H(T_i) = O(H(T_j))$. We call the representatives of the

equivalence classes *basic tiles*, and denote them by $\{\mathcal{T}_1, \ldots, \mathcal{T}_n\}$. By this definition, we can also think of H as a set of dissection rules on the basic tiles, and extend it to collections of tiles as before. For a tile T in the tiling we say that T is *of type i* if it is equivalent to $\mathcal{T}_i$.

Proposition 2.3. *If H is a primitive substitution then $X_H \neq \emptyset$ and for every $\tau \in X_H$ and for every $m \in \mathbb{N}$ there exists a tiling $\tau_m \in X_H$ that satisfies $(\xi H)^m(\tau_m) = \tau$.*

Proof. See [Ro04]. □

Given a tiling $\tau = \tau_0 \in X_H$, for every $m \in \mathbb{N}$ we fix a tiling τ_m as in Proposition 2.3. τ_m is called the *m'th inflation of τ*.

Definition 2.4. A subset $Y \subseteq \mathbb{R}^d$ is called *relatively dense* if there is an $R > 0$ such that $dist(x, Y) \leq R$ for every $x \in \mathbb{R}^d$. It is *uniformly discrete* if there is an $r > 0$ such that for every $y_1 \neq y_2 \in Y$ we have $dist(y_1, y_2) \geq r$. Y is a *separated net* if it is relatively dense and uniformly discrete.

A tiling of $\mathbb{R}^d$ by finitely many tiles gives rise to a relatively dense set by placing one point in each tile. Moving each point a bounded distance in its corresponding tile one also gets a separated net, and it is easy to see that any two such nets are biLipschitz equivalent to each other. That is, there is a bijective map ϕ between them such that ϕ and ϕ^{-1} are both Lipschitz maps. Giving a tiling, we fix one separated net as above and denote it by Y. As we mentioned in the introduction, there is a deep connection between the question whether Y is biLipschitz equivalent to $\mathbb{Z}^d$ and the question that we study here.

We use $|T|$ to denote the area of T (defined by the Lebesgue measure) and $\#F$ for the number of elements in a finite set F. We use the separated net Y, that corresponds to a tiling, to count the number of tiles in a patch P, since $\#(supp(P) \cap Y) = \#\{T \in \tau : T \in P\}$. For short we denote this quantity by $\#(P \cap Y)$.

Lemma 2.5. *Let τ be a primitive substitution tiling of $\mathbb{R}^d$, then there are positive constants $\kappa < \xi^d, c, \alpha$, that depends only on τ, such that for every m and $T \in \tau_m$, we have*

$$|\#(T \cap Y) - \alpha \cdot |T|| \leq c \cdot \kappa^m$$

Proof. See [S11], Lemma 4.3. □

Note that if $T \in \tau_m$ is of type i then $|T| = (\xi^d)^m |\mathcal{T}_i|$. Then we have the following immediate corollary:

Corollary 2.6. *Let τ be a primitive substitution tiling of $\mathbb{R}^d$, then there are positive constants $\eta < 1, c, \alpha$, that depends only on τ, such that for every m and $T \in \tau_m$, we have*

(2) $$\max\left\{\frac{|\#(T \cap Y) - \alpha \cdot |T||}{\alpha \cdot |T|}, \frac{|\#(T \cap Y) - \alpha \cdot |T||}{\#(T \cap Y)}\right\} \leq c \cdot \eta^m.$$

The number α is the *asymptotic density of Y*.

3 Proof of the Main Theorem

In this section we prove Theorem 1.2. We denote by $D_\phi(x)$ the derivative of $\phi : \mathbb{R}^d \to \mathbb{R}^d$ at the point x, and we use $Jac(\phi)$ to denote the Jacobian of ϕ, $Jac(\phi)(x) = \det(D_\phi(x))$.

Proof of Theorem 1.2. First note that it is enough to find a biLipschitz homeomorphism Φ' with $Jac(\Phi') = \beta \cdot f_\tau$ for some positive constant β. Then $\Phi = \beta^{-1/d} \cdot \Phi'$ is as required.

For every $m \geq 1$ every tile $T \in \tau_m$ is tiled with tiles of τ_{m-1}. Define $f_m^T : T \to \mathbb{R}$ to be the average of f_τ on T', on every $T' \in \tau_{m-1}$:

$$f_m^T(x) = \begin{cases} \frac{\int_{T'} f_\tau}{|T'|}, & x \in int(T'), \ T' \subseteq T, \ T' \in \tau_{m-1} \\ 0, & \text{otherwise} \end{cases}$$

Obviously f_m^T is a weight function, then by the assumption, for every m and $T \in \tau_m$ there exists a biLipschitz homeomorphism $\varphi_m^T : T \to T$ that satisfies (1). Gluing these homeomorphisms along the boundaries of the tiles of τ_m gives a biLipschitz homeomorphism $\varphi_m : \mathbb{R}^d \to \mathbb{R}^d$ that satisfies

(3)
$$Jac(\varphi_m)(x) \overset{a.e.}{=} \left\{ \frac{|T|}{\int_T f_m} \cdot f_m(x), \quad x \in int(T), \ T \in \tau_m \right.$$

$$biLip(\varphi_m) \leq \left(\frac{\max f_m}{\min f_m} \right)^C,$$

where $f_m : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$f_m(x) = \begin{cases} \frac{\int_{T'} f_\tau}{|T'|}, & x \in int(T'), \ T' \in \tau_{m-1} \\ 0, & \text{otherwise} \end{cases} .$$

Define $\phi_n : \mathbb{R}^d \to \mathbb{R}^d$ by $\phi_n = \varphi_n \circ \varphi_{n-1} \circ \ldots \circ \varphi_1$. We claim that (ϕ_n) has a subsequence that converges uniformly to the desired Φ.

First note that for any patch P we have $\int_P f_\tau = \#\{S \in \tau_0 : S \subseteq P\} = \#(P \cap Y)$. Combining this with (2) we obtain

$$\begin{cases} \alpha^{-1} \max f_m \leq \max\limits_{T \in \tau_{m-1}} \left\{ \frac{\#(T \cap Y)}{\alpha|T|} \right\} \leq 1 + c\eta^m \\ \frac{\alpha}{\min f_m} \leq \max\limits_{T \in \tau_{m-1}} \left\{ \frac{\alpha|T|}{\#(T \cap Y)} \right\} \leq 1 + c\eta^m \end{cases} \implies \frac{\max f_m}{\min f_m} \leq (1 + c\eta^m)^2.$$

Then for every n we have

$$biLip(\phi_n) \leq \prod_{m=1}^n \left(\frac{\max f_m}{\min f_m} \right)^C \leq \left(\prod_{m=1}^\infty 1 + c \cdot \eta^m \right)^{2C}.$$

Since

$$\log \left(\prod_{m=1}^\infty 1 + c \cdot \eta^m \right) \leq \sum_{m=1}^\infty \log (1 + c \cdot \eta^m) \leq \sum_{m=1}^\infty (c \cdot \eta^m) = \frac{c\eta}{1 - \eta},$$

we have a uniform bound for $biLip(\phi_n)$. Then by the Arzela Ascoli we get a biLipschitz homeomorphism $\Phi : \mathbb{R}^d \to \mathbb{R}^d$, with $biLip(\Phi) \leq e^{2C \cdot c \cdot \eta/(1-\eta)}$.

Regarding $Jac(\Phi)$, for any $T \in \tau_m$

$$(4) \qquad \int_T f_m = \sum_{\substack{T' \subseteq T \\ T' \in \tau_{m-1}}} \frac{1}{|T'|} \int_{T'} \#(T' \cap Y) = \#(T \cap Y).$$

By (3), for every $T \in \tau_m, T' \in \tau_{m-1}$ and for a.e. $x \in T' \subseteq T$ we have

$$(5) \qquad Jac(\varphi_m)(x) = \frac{|T|}{\int_T f_m} \cdot f_m(x) = \frac{|T|}{\#(T \cap Y)} \cdot \frac{\#(T' \cap Y)}{|T'|}.$$

Take $x \in \mathbb{R}^d$, denote by T_m the tile of τ_m that contains x, and suppose that $x \in int(T_0)$. Since for every m the map φ_m maps every tile of τ_m to itself, and by (2), we have

$$Jac(\phi_n)(x) = Jac(\varphi_n)(\varphi_{n-1} \circ \ldots \circ \varphi_1(x)) \cdot Jac(\varphi_{n-1})(\varphi_{n-2} \circ \ldots \circ \varphi_1(x)) \cdot \ldots \cdot Jac(\varphi_1)(x) \overset{(5)}{=}$$

$$\frac{|T_n| \cdot \#(T_{n-1} \cap Y)}{\#(T_n \cap Y) \cdot |T_{n-1}|} \cdot \ldots \cdot \frac{|T_1| \cdot \#(T_0 \cap Y)}{\#(T_1 \cap Y) \cdot |T_0|} = \frac{|T_n|}{\#(T_n \cap Y)} \cdot \frac{1}{|T_0|} \xrightarrow{n \to \infty} \frac{\alpha^{-1}}{|T_0|} = \alpha^{-1} f_\tau(x).$$

$\square$

4 The Star-Shaped Lemma

The purpose of this section is to obtain a homeomorphism between star-shaped domains in the plane, with Jacobian 1 a.e. This result will be used in the next section.

Definition 4.1. $T \subseteq \mathbb{R}^d$ is a *star-shaped domain* if there exists a point $p \in int(T)$ such that for every point $x \in T$ the interval between p and x is contained in T, that is $\{tp + (1-t)x : t \in [0,1]\} \subseteq T$. We denote $\langle T, p \rangle$ a star-shaped domain T with a point p as above. For short, we say that *p sees all of T* for this property. Given a star-shaped domain $\langle T, p \rangle$ in $\mathbb{R}^2$, and assume that $p = 0$, every $\theta \in [0, 2\pi]$ defines a *θ-sector of T* in polar coordinates by:
$$T^{(\theta)} = \{(r, \alpha) \in T : \alpha \in [0, \theta]\}.$$

Lemma 4.2. *Suppose $\langle T_1, p_1 \rangle, \langle T_2, p_2 \rangle$ are two star-shaped domains in the plane, with piecewise differentiable boundary and with the same area, then there is a unique homeomorphism $\psi : T_1 \to T_2$, such that:*

- *$\psi(p_1) = p_2$.*

- *ψ maps ∂T_1 injectively onto ∂T_2.*

- *ψ maps every sector of T_1 to a sector of T_2 with the same area.*

- *$Jac(\psi) = 1$ a.e.*

Proof. It suffices to show that there is such a homeomorphism between the unit ball $B = B(0,1)$ and another star-shaped domain $\langle T, 0 \rangle$ with the same area.
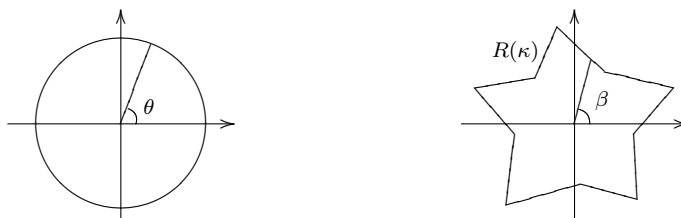
Define a mapping $\psi : B \to T$ by

$$\psi(r\cos(\theta), r\sin(\theta)) = r \cdot R(\beta) \cdot (\cos(\beta), \sin(\beta)),$$

where $R(\kappa)$ is the distance between 0 and ∂T in direction $\kappa \in [0, 2\pi]$, and β is an angle that is defined by the following equation:

$$\frac{1}{2} \int_0^\beta R^2(t)dt = \frac{\theta}{2}.$$

Namely, for every θ we choose $\beta = \beta(\theta)$ such that the sector of angle β in T has the same area as the sector of angle θ in B.

It is now left to the reader to check that ψ satisfies the requirements. $\qquad\square$

5 Application for Star-Shaped Substitution Tilings

Definition 5.1. Let τ be a primitive substitution tiling $\mathbb{R}^d$. We say that τ is a *star-shaped substitution tiling* if every tile of τ is star-shaped with a piecewise differentiable boundary.

In this section we prove Proposition 5.2, that shows that the hypothesis of Theorem 1.2 is satisfied for star-shaped substitution tilings of the plane. Proposition 5.2 generalized Proposition 3.2 of [BK02] from dyadic lattice square tiling to any star-shaped substitution tiling. The proof is obtained by repeating the steps of their proof, with some modifications to our case.

Proposition 5.2. *Let $\langle T, p \rangle$ be a star-shaped domain, with a partition to smaller star-shaped domains $\langle T_1, p_1 \rangle, \ldots, \langle T_n, p_n \rangle$. Then there is a constant C_1 such that for every weight function $f : T \to (0, \infty)$ there is a biLipschitz homeomorphism $\varphi : T \to T$ that satisfies (1), with C_1 instead of C.*

For the proof of Proposition 5.2 we need the following definitions:

Definition 5.3. Let $\langle T, p \rangle$ be a star-shaped domain and assume that $|T| = |B|$, where $B = B(0,1)$. Let $\psi : B \to T$ be the homeomorphism from Lemma 4.2. For a given function $f : T \to \mathbb{R}$, we say that f *is constant on the elevation lines of T* if f is constant on (one dimensional) sets of the form $\psi(\partial B(0, r))$. In a similar manner we can define objects like *contraction around p, star-shaped annulus, neighborhood of the boundary*, etc. We will

also use the same terminology for $\psi : S \to T$, where S is a square instead of a ball.
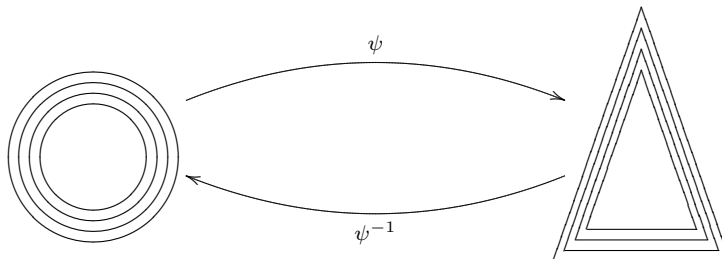
With this terminology we can now state the two lemmas which play the role of Lemma 3.3 and Lemma 3.6 from [BK02].

Lemma 5.4. *Let $\langle T, p \rangle$ be a star-shaped domain. There is a constant C_2 with the following property: Suppose $h_1, h_2 : T \to \mathbb{R}$ are continuous positive functions, which are constant on the elevation lines of T, and $\int_T h_1 = \int_T h_2$. Then there exists a biLipschitz homeomorphism $\phi : T \to T$, which fixes ∂T pointwise, so that $Jac(\phi) = \frac{h_1}{h_2 \circ \phi}$ a.e. and*

$$(6) \qquad biLip(\phi) \leq \left(\frac{\max h_1}{\min h_1} \right)^{C_2} \left(\frac{\max h_2}{\min h_2} \right)^{C_2}.$$

Lemma 5.5. *Let $\langle T, p \rangle$ be a star-shaped domain and let A be a star-shaped annulus that is obtained by removing a contracted copy of T around p. Then there is a constant C_3 such that for every $g_1, g_2 : A \to \mathbb{R}$, positive Lipschitz functions with*

$$\int_A g_1 = \int_A g_2 = |A|,$$

there is a biLipschitz homeomorphism $\phi : A \to A$ with $Jac(\phi) = \frac{g_1}{g_2 \circ \phi}$ a.e., and

$$biLip(\phi) \leq \left[\frac{\max g_1}{\min g_1} (1 + Lip(g_1)) \right]^{C_3} \cdot \left[\frac{\max g_2}{\min g_2} (1 + Lip(g_2)) \right]^{C_3}.$$

Moreover, when $g_1|_{\partial A} = g_2|_{\partial A}$, then ϕ can be chosen to fix ∂A pointwise.

Proof of Lemma 5.4. As in [BK02], the general case follows from the special case $h_2 \equiv 1$, $\int_T h_1 = |T|$. Indeed, setting $\overline{h_i} = (|T| / \int_T h_i)h_i$ we have $\int_T \overline{h_i} = |T|$ for $i \in \{1, 2\}$. Applying the result of the special case we get biLipschitz homeomorphisms $\phi_1, \phi_2 : T \to T$ with $Jac(\phi_i) = \overline{h_i}$, and

$$biLip(\phi_i) \leq \left(\frac{\max \overline{h_i}}{\min \overline{h_i}} \right)^{C_2} = \left(\frac{\max h_i}{\min h_i} \right)^{C_2}.$$

Then $\phi = \phi_2^{-1} \circ \phi_1$ satisfies $Jac(\phi)(x) = Jac(\phi_2^{-1})(\phi_1(x)) \cdot Jac(\phi_1)(x) = \frac{\overline{h_1}(x)}{\overline{h_2}(\phi_2^{-1} \circ \phi_1(x))} = \frac{h_1}{h_2 \circ \phi}(x)$ a.e., and $biLip(\phi)$ satisfies (6).

For that special case, we denote $h := h_1$, and assume without loss of generality that $|T| = |S_1|$, where $S_r = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_\infty \leq r \right\}$, and $\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_\infty = \max\{|x|, |y|\}$. Let $\psi :$

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$S_1 \to T$ be as in Lemma 4.2. Denote by $f = h \circ \psi$, then $f : S_1 \to \mathbb{R}$ satisfies the conditions of Lemma 3.3 from [BK02]. Define $g : [0, 1] \to [0, 1]$, and a biLipschitz homeomorphism $\widetilde{\phi} : S_1 \to S_1$, by

$$g(r) = \frac{1}{2}\sqrt{\int_{S_r} f}, \qquad \text{and} \qquad \widetilde{\phi}(x) = g(\|x\|_\infty) \cdot \frac{x}{\|x\|_\infty}.$$

It was proved in [BK02, proof of Lemma 3.3] that

$$(7) \qquad Jac(\widetilde{\phi}) = f \text{ a.e. }, \quad biLip(\widetilde{\phi}) \leq k_1 \frac{\max f}{\min f} = k_1 \frac{\max h}{\min h},$$

and, when $\frac{\max h}{\min h}$ is close to 1

$$(8) \qquad \left\| D_{\widetilde{\phi}} - I \right\| \leq k_2 \left(\frac{\max f}{\min f} - 1 \right) = k_2 \left(\frac{\max h}{\min h} - 1 \right),$$

where k_1 and k_2 are independent of h.

Define $\phi = \psi \circ \widetilde{\phi} \circ \psi^{-1} : T \to T$. Then ϕ is a biLipschitz homeomorphism that satisfies:

$$Jac(\phi)(x) = \underbrace{Jac(\psi)(\widetilde{\phi} \circ \psi^{-1}(x))}_{=1} \cdot \underbrace{Jac(\widetilde{\phi})(\psi^{-1}(x))}_{=f(\psi^{-1}(x))} \cdot \underbrace{Jac(\psi^{-1})(x)}_{=1} = h(x) \text{ a.e.}$$

By (7) and (8), there are C, C', that depend on ψ, such that

$$(9) \qquad biLip(\phi) \leq biLip(\psi) \cdot biLip(\widetilde{\phi}) \cdot biLip(\psi^{-1}) \leq k_1 \cdot C \cdot \frac{\max f}{\min f} = k_1' \cdot \frac{\max h}{\min h},$$

and when $\frac{\max h}{\min h}$ is close to 1,

$$\|(D_\phi - I)(x)\| = \left\| D_{\psi \circ (\widetilde{\phi} - I) \circ \psi^{-1}}(x) \right\| \leq C' \cdot \left\| (D_{\widetilde{\phi}} - I)(\psi^{-1}(x)) \right\| = k_2' \left(\frac{\max h}{\min h} - 1 \right),$$

where k_1' and k_2' do not depend on h. This implies that

$$(10) \qquad \left\| D_\phi^{\pm 1} \right\| \leq \left(\frac{\max h}{\min h} \right)^{k_2''},$$

where k_2'' does not depend on h. Combining (9) and (10) we get

$$biLip(\phi) \leq \left(\frac{\max h}{\min h} \right)^{C_2},$$

as required. $\qquad \qquad \square$

The proof of Lemma 5.5 is obtained in a similar way by following the proof of Lemma 3.6 of [BK02], and using ψ from Lemma 4.2 as we did in the proof of Lemma 5.4.

Finally, before we approach the proof of Proposition 5.2 we need the following claim:

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Claim 5.6. *Let $\langle T, p \rangle$ be a star-shaped domain, with a partition to smaller star-shaped domains $\langle T_1, p_1 \rangle, \ldots, \langle T_n, p_n \rangle$. For an $r \in (0,1)$ we denote by T_i^r the contraction of T_i by r around p_i. Then there exists an $r > 0$ and a point $q \in int(T) \setminus \bigcup T_i^r$ such that for every $i \in \{1, \ldots, n\}$ and $x \in T_i^r$, the interval between q and x is contained in T.*

Proof. We say that "q sees y" if $\{tq + (1-t)y : t \in [0,1]\} \subseteq T$, and write

$$T_{1-\varepsilon} = \{x \in T : d(x, \partial T) \geq \varepsilon\}.$$

We first show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $q \in B(p, \delta)$ and $y \in T_{1-\varepsilon}$, q sees y. Assume otherwise, given an $\varepsilon > 0$, for every $n \in \mathbb{N}$ there is a $q_n \in B(p, \frac{1}{n})$ and a point $y_n \in T_{1-\varepsilon}$ such that q_n does not see y_n. That is, for every n there is a $t_n \in [0,1]$ such that $z_n = t_n q_n + (1 - t_n) y_n \notin T$. We know that $q_n \xrightarrow{n \to \infty} p$, and by passing to subsequences we may assume that $t_n \xrightarrow{n \to \infty} t \in [0,1]$ and $y_n \xrightarrow{n \to \infty} y \in T_{1-\varepsilon}$. Then $z_n \xrightarrow{n \to \infty} z = tp + (1-t)y \notin int(T)$. We have obtained a point $y \in T_{1-\varepsilon}$, in particular $y \in int(T)$, such that the interval between p and y travels out of $int(T)$, a contradiction.

Since $p_1, \ldots, p_n \in int(T)$, we may fix $r_1, \varepsilon > 0$ such that for all $0 < r < r_1$ we have $\bigcup T_i^r \subseteq T_{1-\varepsilon}$. Let $\delta > 0$ be as above, so there is an $r_2 > 0$ such that the union $\bigcup T_i^{r_2}$ does not cover all of $B(p, \delta)$. So for $r = \min\{r_1, r_2\}$, any point $q \in B(p, \delta) \setminus \bigcup T_i^r$ is as required. $\qquad\square$

We are now ready to prove Proposition 5.2. The proof follows the steps of the proof of Proposition 3.2 in [BK02], replacing Lemmas 3.3 and 3.6 there by Lemmas 5.4 and 5.5 from above. For the convenience of the reader we repeat their proof, in our context.

Proof of proposition 5.2. Let $r > 0$ and let $q \in int(T) \setminus \bigcup T_i^r$ be as in Claim 5.6. Define

$$S = \{y \in T : q \text{ sees } y\},$$

and let A be the star-shaped annulus that is obtained by removing from S a contracted copy of S around q, S', such that $S \setminus S'$ still contains $\bigcup T_i^r$.

We may assume that $\int_T f = |T|$. Define the following functions:
Let $f_2 : T \to (0, \infty)$ be a Lipschitz function such that $f_2 = \min f$ on the complement of $\bigcup T_i^r$, and is constant on the elevation lines of each of the T_i's. In addition

$$\int_{T_i} f_2 = \int_{T_i} f, \quad \frac{\max f_2}{\min f_2} \leq \left(\frac{\max f}{\min f} \right)^{k_1}, \quad \text{and} \quad 1 + Lip(f_2) \leq \left(\frac{\max f}{\min f} \right)^{k_1},$$

where k_1 is independent of f.
Let $f_3 : T \to (0, \infty)$ be a Lipschitz function such that $f_3 = \min f$ on the complement of A, f_3 is constant on the elevation lines of S, and

$$\int_T f_3 = \int_T f, \quad \frac{\max f_3}{\min f_3} \leq \left(\frac{\max f}{\min f} \right)^{k_2}, \quad \text{and} \quad 1 + Lip(f_3) \leq \left(\frac{\max f}{\min f} \right)^{k_2},$$

where k_2 is independent of f.
Finally, set $f_4 = 1$.

Since for every $i \in \{1, \ldots, n\}$ we have $\int_{T_i} f_2 = \int_{T_i} f$, we can apply Lemma 5.4 on $f|_{T_i}$ and $f_2|_{T_i}$, separately for every i, and get a biLipschitz homeomorphism $\psi_1^i : T_i \to T_i$ with $Jac(\psi_1^i) = \frac{f}{f_2 \circ \psi_1^i}$ a.e. Gluing these homeomorphisms along the boundaries of the T_i's we obtain a biLipschitz homeomorphism $\psi_1 : T \to T$ with

$$Jac(\psi_1) = \frac{f}{f_2 \circ \psi_1} \text{ a.e.} \quad \text{and} \quad biLip(\psi_1) \leq \left(\frac{\max f}{\min f}\right)^{C_2 k_1}.$$

Since for every $x \in T \smallsetminus A$ we have $f_2(x) = f_3(x) = \min f$, $\int_A f_2 = \int_A f_3 \geq |A|$. Define $\bar{f}_j = \left(|A| / \int_A f_j\right) \cdot f_j$, for $j \in \{2, 3\}$, so $Lip(\bar{f}_j) \leq Lip(f_j)$. Applying Lemma 5.5 to the star-shaped annulus A, with $\bar{f}_2$ and $\bar{f}_3$, we get a biLipschitz homeomorphism $\bar{\psi}_2 : A \to A$, that fixes ∂A pointwise, with

$$(11) \qquad Jac(\bar{\psi}_2) = \frac{\bar{f}_2}{\bar{f}_3 \circ \bar{\psi}_2} = \frac{f_2}{f_3 \circ \bar{\psi}_2} \text{ a.e.} \quad \text{and} \quad biLip(\bar{\psi}_2) \leq \left(\frac{\max f}{\min f}\right)^{C_3 k_2}.$$

We can extend $\bar{\psi}_2$ to $\psi_2 : T \to T$ by defining it to be the identity outside of A, and we get a biLipschitz homeomorphism of T, satisfying (11) (since outside of A we get $\frac{f_2}{f_3 \circ \psi_2}(x) = 1$).

Finally, we apply Lemma 5.4 again on f_3 and f_4, to get a biLipschitz homeomorphism $\psi_3 : T \to T$ with

$$Jac(\psi_3) = \frac{f_3}{f_4 \circ \psi_3} \text{ a.e.} \quad \text{and} \quad biLip(\psi_3) \leq \left(\frac{\max f}{\min f}\right)^{C_2 k_2}.$$

Now define $\phi = \psi_3 \circ \psi_2 \circ \psi_1$. So $C_1 = C_2 k_1 + C_3 k_2 + C_2 k_2$ satisfies the statement of the proposition, and we have

$$Jac(\phi)(x) = Jac(\psi_3)(\psi_2 \circ \psi_1(x)) \cdot Jac(\psi_2)(\psi_1(x)) \cdot Jac(\psi_1)(x) =$$

$$\frac{f_3}{f_4 \circ \psi_3}(\psi_2 \circ \psi_1(x)) \cdot \frac{f_2}{f_3 \circ \psi_2}(\psi_1(x)) \cdot \frac{f}{f_2 \circ \psi_1}(x) = \frac{f}{f_4 \circ \phi}(x) = f(x)$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Theorem 1.4 is a direct consequence of Theorem 1.2 and Proposition 5.2. Corollary 1.5 follows directly from Theorem 1.4.

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