PERIODICITY OF JOINT CO-TILES IN $\mathbb{Z}^d$

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Abstract. Given finite subsets $F_1, \dots, F_k$ in $\mathbb{Z}^d$, a joint co-tile is a set $A \subseteq \mathbb{Z}^d$ that satisfies $F_j \oplus A = \mathbb{Z}^d$ for all $1 \leq j \leq k$. We study the structure of joint co-tiles in $\mathbb{Z}^d$. We introduce a notion of independence for a tuple of finite subsets of $\mathbb{Z}^d$. We prove that for any $d \geq 1$, any joint co-tile for d independent sets is periodic. This generalizes a classical result of Newman stating that any tiling of $\mathbb{Z}$ by a finite set is periodic. For a $(d-1)$-tuple of finite subsets of $\mathbb{Z}^d$ that satisfy a certain technical condition that we call property $(\star)$, we prove that any joint co-tile decomposes into disjoint $(d-1)$-periodic sets. Consequently, we show that for a $(d-1)$-tuple of finite subsets of $\mathbb{Z}^d$ that satisfy property $(\star)$, the existence of a joint co-tile implies the existence of periodic joint co-tile. These results are generalizations to higher dimensions of Bhattacharya's theorem (the proof of the periodic tiling conjecture for $\mathbb{Z}^2$) and Greenfeld-Tao's theorem about the structure of co-tiles in $\mathbb{Z}^2$. Conversely, we prove that if a finite subset F in $\mathbb{Z}^d$ admits a periodic co-tile A, then there exist $(d-1)$ additional tiles that together with F are independent and admit A as a joint co-tile, and $(d-2)$ additional tiles that together with F satisfy the property $(\star)$. Combined, our results give a new necessary and sufficient condition for a finite subset of $\mathbb{Z}^d$ to tile periodically. We also discuss tilings and joint tilings in other countable abelian groups.

1. Introduction

For a countable abelian group Γ we write $F \Subset \Gamma$ to indicate that F is a finite subset of Γ. For $A \subseteq \Gamma$ we denote by

$$F \oplus A = \biguplus_{a \in A} (F + a),$$

where the notation of the right-hand side stands for a disjoint union of the sets $\{F + a\}_{a \in A}$. The notation $F \oplus A = E$ thus means that every $e \in E$ has a unique representation as $e = f + a$ where $f \in F$ and $a \in A$. We say that F *tiles* Γ if there exists a collection of disjoint union of translates of F whose union is equal to Γ. That is, F tiles Γ if there exists a set $A \subseteq \Gamma$ such that

$$F \oplus A = \Gamma. \tag{1}$$

In that case, we say that A is a *co-tile* for the *tile* F. Let $g, h : \Gamma \to \mathbb{R}$, where Γ is a countable abelian group. We denote by $g * h$ the usual convolution function given by

$$g * h(x) = \sum_{y \in \Gamma} g(y) \cdot h(x - y).$$

Using this notation, equation (1) is equivalent to $\mathbf{1}_F * \mathbf{1}_A = 1$, where $\mathbf{1}_X$ denotes the indicator function of the set X.

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Let Γ be a countable abelian group. Elements $g_1, \ldots, g_k \in \Gamma$ are called *independent* if the only integers $n_1, \ldots, n_k \in \mathbb{Z}$ that satisfy $\sum_{j=1}^{k} n_j g_j = 0$ are $n_1 = \ldots = n_k = 0$. Recall that the rank of an abelian group is the maximal size of an independent set.

Suppose that Γ is an abelian group of rank d and that $k \leq d$. A set $C \subseteq \Gamma$ is called *k-periodic* if there exists a subgroup $L \leq \Gamma$, with $\mathrm{rank}(L) \geq k$, such that $C + L = C$. In the case that $k = d$ we will also say that C is *periodic* instead of d-periodic. We say that a tile set $F \Subset \Gamma$ *tiles* Γ *periodically* if there exits a periodic co-tile for F. If F tiles Γ but does not admit a periodic co-tile, then the set F is called *aperiodic*.

Newman [New77] proved that any tiling of $\Gamma = \mathbb{Z}$ by a finite set is periodic. Already for $\Gamma = \mathbb{Z}^2$, it is not difficult to find tilings of Γ by a finite set that are not even 1-periodic. See [GT21a, §1.3] for some examples and a brief discussion. Still, it is natural to ask for different generalizations of Newman's theorem to higher-rank abelian groups. It has been conjectured for some time that for any $F \Subset \mathbb{Z}^d$, if there exists $A \subseteq \mathbb{Z}^d$ such that $F \oplus A = \mathbb{Z}^d$ then there exists a periodic $A' \subseteq \mathbb{Z}^d$ such that $F \oplus A' = \mathbb{Z}^d$ [LW96], [GS87]. This conjecture became known as the *periodic tiling conjecture*. The periodic tiling conjecture can be interpreted as an attempt to generalize Newman's theorem. The $\mathbb{Z}^2$ case of the periodic tiling conjecture was proved several years ago by Bhattacharya [Bha20]. Other instances of the periodic tiling conjecture have been proved, under additional assumptions [BN91, Khe22, Ken92, Sze98, WvL84]. The periodic tiling conjecture has recently been disproved for sufficiently large d by Greenfeld and Tao [GT22].

In this paper, we study the structure of sets $A \subseteq \mathbb{Z}^d$ that satisfy

$$F_j \oplus A = \mathbb{Z}^d \text{ for all } j = 1, \ldots, k, \tag{2}$$

for subsets $F_1, \ldots, F_k \Subset \mathbb{Z}^d$. We refer to such an A as a *joint co-tile* for $F_1, \ldots, F_k$. In [GT21b], sets $A \subseteq \mathbb{Z}^d$ satisfying (2) have been referred to as solutions to the *system of tiling equations*. As with ordinary systems of linear equations, it makes sense to introduce a notion of independence in this setup. For $F \Subset \mathbb{Z}^d$ we denote

$$F^* := F \setminus \{0\}.$$

We say that $(F_1, \ldots, F_k)$ is an *independent tuple* of tiles (or k *independent tiles*) if each F_j is a finite subset of $\mathbb{Z}^d$, with $0 \in F_j$, and for every choice of $v_1 \in F_1^*, \ldots, v_k \in F_k^*$, the k-tuple $(v_1, \ldots, v_k)$ is independent (equivalently here, linearly independent vectors over $\mathbb{Q}$, or similarly over $\mathbb{R}$ or $\mathbb{C}$). Notice that if $(F_1, \ldots, F_k)$ is an independent tuple of tiles then $k \leq d$. Observe that the existence of a joint co-tile for $F_1, \ldots, F_k \Subset \mathbb{Z}^d$ implies that $|F_1| = |F_2| = \ldots = |F_k|$ (see Proposition 2.5).

Building on methods developed in [Bha20], [GT21a] and earlier work, we prove the following:

Theorem 1.1. *For every $1 \leq k \leq d$, the indicator function of any joint co-tile for k independent tiles in $\mathbb{Z}^d$ is equal, up to a constant, to a sum of $[0,1]$-valued k-periodic functions.*

The case $k = 1$ of Theorem 1.1 was proven in [GT21a]. As a consequence of Theorem 1.1, we obtain the following generalization of Newman's result for any dimension:

Theorem 1.2. *Any joint co-tile for d independent tiles in $\mathbb{Z}^d$ is d-periodic. Furthermore, if $\mathbf{1}_{F_i} * f = 1$ holds for d independent tiles $(F_1, \ldots, F_d)$ and a bounded function $f : \mathbb{Z}^d \to \mathbb{Z}$, then f is d-periodic.*

We discuss further generalizations of Newman's theorem in Section 4 and particularly to the group $\mathbb{Z} \times (\mathbb{Z}/p\mathbb{Z})$ in Proposition 4.3.

We say that a set $A \subseteq \mathbb{Z}^d$ is *piecewise k-periodic* if there exist $A_1, \dots, A_r \subset \mathbb{Z}^d$ such that $A = \biguplus_{j=1}^r A_j$ and each A_j is k-periodic. Note that [Bha20] and [GT21a] used *weakly periodic* for piecewise 1-periodic. In [GT21a] it was shown that any $A \subseteq \mathbb{Z}^2$ satisfying $F \oplus A = \mathbb{Z}^2$ is piecewise 1-periodic, whereas in [Bha20] it was shown that almost every solution to $F \oplus A = \mathbb{Z}^2$ is piecewise 1-periodic, with respect to any invariant measure on the space of solutions. The apriori weaker "almost everywhere" result sufficed to prove the $\mathbb{Z}^2$ periodic tiling conjecture. The following result shows that the existence of piecewise $(d-1)$-periodic joint co-tiles implies the existence of d-periodic joint co-tiles. For $k = 1$ and $d = 2$ it coincides with the results in [Bha20], [GT21a], deducing 2-periodicity from piecewise 1-periodicity.

Theorem 1.3. *Let k and d be positive integers and let $F_1, \dots, F_k \Subset \mathbb{Z}^d$. If $F_1, \dots, F_k$ admit a piecewise $(d-1)$-periodic joint co-tile, then they admit a d-period joint co-tile.*

We now define an additional condition on a tuple of tiles, that is needed for the formulation of a certain generalization of Bhattacharya's and Greenfeld-Tao's theorems to $d > 2$:

Definition 1.4. Let $(F_1, \dots, F_{d-1})$ be a tuple of tiles in $\mathbb{Z}^d$, $d \geq 2$. We say that $(F_1, \dots, F_{d-1})$ has *property* $(\star)$ if it is an independent tuple and for every $(v_1, \dots, v_{d-1}), (w_1, \dots, w_{d-1}) \in F_1^* \times \dots \times F_{d-1}^*$ such that

$$\operatorname{span}(v_1, \dots, v_{d-1}) = \operatorname{span}(w_1, \dots, w_{d-1}),$$

we have $v_i = w_i$ for all $1 \leq i \leq d-2$.

Theorem 1.5. *Let $(F_1, \dots, F_{d-1})$ be a tuple of tiles in $\mathbb{Z}^d$ that has property $(\star)$. Then any joint co-tile for $F_1, \dots, F_{d-1}$ is piecewise $(d-1)$-periodic.*

The next statement follows immediately from Theorem 1.5 together with Theorem 1.3.

Corollary 1.6. *Let $(F_1, \dots, F_{d-1})$ be a tuple of tiles in $\mathbb{Z}^d$ that has property $(\star)$. If $(F_1, \dots, F_{d-1})$ admits a joint co-tile then it admits a d-periodic joint co-tile.*

Note that for $d = 2$, property $(\star)$ is vacuous, hence Theorem 1.5 reduces to the statement that any co-tile for a finite subset of $\mathbb{Z}^2$ is piecewise 1-periodic (Greenfeld-Tao's theorem) and Corollary 1.6 reduces to the statement that any finite subset of $\mathbb{Z}^2$ that admits a co-tile also admits a periodic co-tile (Bhattacharya's theorem). Hence for $d \geq 3$, it is natural to ask whether property $(\star)$ is a necessary condition for the existence of a periodic joint co-tile of $(d-1)$ tiles of $\mathbb{Z}^d$.

We note a particular application of our methods, although not directly related to our main results:

Theorem 1.7. *Suppose that $\mathbb{Z}^d$ decomposes into $(d-1)$-periodic subsets $A_1, \dots, A_r \subset \mathbb{Z}^d$, where at least one of them is not d-periodic. Then there exists $\Gamma \leq \mathbb{Z}^d$ of rank $d-1$ so that $A_j + \Gamma = A_j$ for all $1 \leq j \leq r$.*

On the other hand, we obtain the following converse results for Theorem 1.2 and Corollary 1.6.

Theorem 1.8. *Suppose that $\{0\} \subsetneq F \Subset \mathbb{Z}^d$ admits a periodic tiling $A \subseteq \mathbb{Z}^d$, then there exist $F_1, \dots, F_{d-1} \Subset \mathbb{Z}^d$ with $0 \in F_j$ and $F_j \oplus A = \mathbb{Z}^d$ for all $1 \leq j \leq d$, such that*

(a) $(F_1, \dots, F_{d-1}, F)$ is a d-tuple of independent tiles.
(b) $(F_1, \dots, F_{d-2}, F)$ has property $(\star)$.

Combining Corollary 1.6 and Theorem 1.8 (b) we obtain the following:

Corollary 1.9. *A finite set* $\{0\} \subsetneq F \Subset \mathbb{Z}^d$ *tiles* $\mathbb{Z}^d$ *periodically if and only if there exists* $F_1, \ldots, F_{d-2} \Subset \mathbb{Z}^d$ *and* $A \subset \mathbb{Z}^d$ *such that* $(F_1, \ldots, F_{d-2}, F)$ *has property* $(\star)$*,* $F \oplus A = \mathbb{Z}^d$ *and* $F_j \oplus A = \mathbb{Z}^d$ *for all* $1 \leq j \leq d-2$*.*

Remark 1.10. Note that F and A play a symmetric role in the equation $F \oplus A = \mathbb{Z}^d$, A is a co-tile for F, but F is also a co-tile for A. Assuming that $F \Subset \mathbb{Z}^d$ and that $F \oplus A = \mathbb{Z}^d$, the periodic tiling conjecture asks about a specific property of the set of co-tiles of F. In view of Corollary 1.9, that property is equivalent to a property of the set of co-tiles of A. In particular for $d = 3$, let $F \Subset \mathbb{Z}^3$, $A \subset \mathbb{Z}^3$ such that $F \oplus A = \mathbb{Z}^3$. Then F tiles $\mathbb{Z}^3$ periodically if and only if there is another co-tile F' for A such that (F', F) has property $(\star)$.

The structure of the paper is as follows. Section 2 contains basic background and definitions. In Section 3 we prove Theorem 3.1, a periodic decomposition theorem for joint co-tiles, which is a refinement of Theorem 1.1. From Theorem 3.1, we directly deduce Theorem 1.1 and Theorem 1.2. In Section 4, we discuss generalizations of Theorem 3.1, Theorem 1.1 and Theorem 1.2 to countable abelian groups. This allows us to extend Newman's Theorem to tilings of the group $\mathbb{Z} \times (\mathbb{Z}/p\mathbb{Z})$. In Section 5 we prove Theorem 1.5, which asserts that property $(\star)$ implies piecewise $(d-1)$-periodicity of joint co-tiles. Then in Section 6 we prove Theorem 1.7 and deduce Theorem 1.3. Section 7 is dedicated to the proof of Theorem 1.8. Finally, Section 8 contains concluding remarks and related questions.

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2. Preliminaries

A function $f : \mathbb{Z}^d \to \mathbb{R}$ is called L-periodic, where $L \leq \mathbb{Z}^d$, if for every $x \in \mathbb{Z}^d$ and $v \in L$ we have $f(x+v) = f(x)$. Recall that a set $A \subseteq \mathbb{Z}^d$ is *piecewise k-periodic* if A is the disjoint union of k-periodic sets.

Definition 2.1. Let Γ_1, Γ_2 be abelian groups. For $f : \Gamma_1 \to \Gamma_2$ and $v \in \Gamma_1$, we define the *discrete derivative of f in direction v*, $D_v f : \Gamma_1 \to \Gamma_2$, by

$$D_v f(w) := f(w) - f(w - v).$$

A function $P : \Gamma_1 \to \Gamma_2$ is called *a polynomial map of degree at most r* if

$$\forall\, v_1, \ldots, v_{r+1} \in \Gamma_1 : \quad D_{v_1} \ldots D_{v_{r+1}} P = 0$$

(where for consistency $P \equiv 0$ is a polynomial of degree -1). Given a subgroup $\Gamma_3 < \Gamma_1$, we say that $P : \Gamma_1 \to \Gamma_2$ is a *polynomial map of degree at most r with respect to* Γ_3 if

$$\forall\, v_1, \ldots, v_{r+1} \in \Gamma_3 : \quad D_{v_1} \ldots D_{v_{r+1}} P = 0.$$

The following basic facts about polynomials will be useful for us. Lemma 2.2 below is due to Leibman [Lei02, Prop. 1.21]. We include a short proof for the reader's convenience.

Lemma 2.2. *Let* $P : \mathbb{Z}^d \to \mathbb{R}$ *be a polynomial map with respect to a finite index subgroup* $L \leq \mathbb{Z}^d$*, which is bounded, then* P *is constant on cosets of* L*.*

Proof. Let $r \in \mathbb{N}$ denote the degree of P, as a polynomial with respect to L. It is clear from Definition 2.1 that if r is equal to 0, then the restriction of P to each coset of L is a constant. Similarly, if the degree of P is equal to 1, then the restriction of P to each coset of L is a constant plus a non-trivial homomorphism (see e.g. [Lei02]). For contradiction, we may assume that $r \geq 1$. Observe that since P is bounded, for every $v \in L$ we have $D_v P \subseteq P(\mathbb{Z}^d) - P(\mathbb{Z}^d)$, thus $D_v P$ is bounded. Therefore, for every $v_1, \ldots, v_{r-1} \in L$ the function $D_{v_1} \ldots D_{v_{r-1}} P$ is a bounded polynomial map of degree exactly one, with respect to L. But non-trivial homomorphisms into $\mathbb{R}$ are unbounded, a contradiction. □

Definition 2.3. We say that a bounded function $f : \mathbb{Z}^d \to \mathbb{R}$ has mean m if

$$\lim_{n\to\infty} \frac{1}{|B_n|} \sum_{v \in B_n} f(v) = m, \tag{3}$$

where $B_n = \{-n, \ldots, n\}^d$.

We say that $f : \mathbb{Z}^d \to \mathbb{R}/\mathbb{Z}$ is equidistributed in $\mathbb{R}/\mathbb{Z}$ if

$$\lim_{n\to\infty} \frac{1}{|B_n|} \sum_{v \in B_n} g(f(v)) = \int_0^1 g(x)dx \tag{4}$$

holds for every continuous function $g : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$, where we identify $g : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ with $g : \mathbb{R} \to \mathbb{R}$ such that $g(x + 1) = x$ for all $x \in \mathbb{R}$.

We will use the following version of Weyl's equidistribution theorem for multivariate polynomials, see for instance [Yif22].

Theorem 2.4 (Weyl's equidistribution theorem for polynomials in several variables)**.** *Let $P : \mathbb{Z}^d \to \mathbb{R}/\mathbb{Z}$ be a polynomial map with respect to a finite index subgroup Γ of $\mathbb{Z}^d$. Then on every coset $v + \Gamma$ of Γ, the restriction of P to $v + \Gamma$ is either equidistributed in $\mathbb{R}/\mathbb{Z}$ or periodic.*

We implicitly rely on the following basic observation:

Proposition 2.5. *Let $F \Subset \mathbb{Z}^d$. Suppose that $F \subset B_{n_0}$ for some $n_0 \in \mathbb{N}$ and that $f : \mathbb{Z}^d \to \mathbb{R}$ is a bounded function satisfying $\mathbf{1}_F * f = 1$. Denote by $C = |F|(\max f - \min f)$. Then for every $n > n_0$ one has*

$$|B_{n-n_0}| - C\,|B_{n+n_0} \setminus B_{n-n_0}| \leq |F| \sum_{w \in B_n} f(w) \leq |B_{n-n_0}| + C\,|B_{n+n_0} \setminus B_{n-n_0}|\,, \tag{5}$$

and thus the function f has mean $\frac{1}{|F|}$. In particular, if $F_1, F_2 \Subset \mathbb{Z}^d$ satisfy $\mathbf{1}_{F_1} * f = \mathbf{1}_{F_2} * f = 1$, then $|F_1| = |F_2|$.

Proof. Pick $n_0 \in \mathbb{N}$ such that $F \subset B_{n_0}$. Observe that $\mathbf{1}_F * f = 1$ implies that for every $n > n_0$ we have

$$\mathbf{1}_{B_{n-n_0}} - C \cdot \mathbf{1}_{B_{n+n_0} \setminus B_{n-n_0}} \leq \mathbf{1}_F * f|_{B_n} \leq \mathbf{1}_{B_{n-n_0}} + C \cdot \mathbf{1}_{B_{n+n_0} \setminus B_{n-n_0}},$$

where $f|_{B_n}$ denotes the restriction of f to B_n. Taking the sum of the values of these functions over all $z \in \mathbb{Z}^d$ implies that (5) holds for every $n > n_0$. Since $\lim_{n\to\infty} \frac{|B_{n-n_0}|}{|B_n|} = 1$ and $\lim_{n\to\infty} \frac{|B_{n+n_0} \setminus B_{n-n_0}|}{|B_n|} = 0$, dividing (5) by $|F| \cdot |B_n|$ and letting $n \to \infty$ yields the assertion. □

Remark 2.6. The mean of a function $f : \Gamma \to \mathbb{R}$ is defined similarly, using (3), for any countable amenable group Γ, in which case B_n is replaced by a Følner sequence in Γ, and an analogue of Proposition 2.5 holds in this more general context as well. In Section 8, we implicitly apply Proposition 2.5 for countable abelian groups Γ, which are in particular amenable.

2.1. **Shifts of finite type.** The space of co-tiles for a given finite set $F \subset \mathbb{Z}^d$, or more generally, the space of joint co-tiles for a given collection of sets, can naturally be endued with the structure of a compact topological space on which $\mathbb{Z}^d$ acts by homeomorphisms. Topological dynamical systems of this kind are called *$\mathbb{Z}^d$-subshifts*, more specifically *subshifts of finite type*. We include here relevant terminology and basic facts from the field of symbolic dynamics, particularly regarding shifts of finite type. We refer to [LM95] for a comprehensive introduction to symbolic dynamics.

Let Σ be a finite set (alphabet) and Γ a finitely generated abelian group. The set of functions from Γ to Σ, denoted Σ^Γ, is called *the full Γ-shift*. For $x \in \Sigma^\Gamma$ and $v \in \Gamma$, we use x_v to denote the value of x at v (this is an element of Σ). Also for $x \in \Sigma^\Gamma$ and $v \in \Gamma$ we denote by $\sigma_v(x) \in \Sigma^\Gamma$ the *shift of x by v*, which is given by

$$\sigma_v(x)_w = x_{v+w}.$$

Endowing Σ^Γ with the product topology, where the topology on Σ is the discrete topology, makes Σ^Γ a compact Γ-space. A closed, non-empty and Γ-invariant subset $X \subseteq \Sigma^\Gamma$ is called a *Γ-subshift*. For $x \in \Sigma^\Gamma$, the *stabilizer of x* is defined to be

$$\mathrm{stab}(x) = \{v \in \Gamma : \ \sigma_v(x) = x\},$$

which is a (possibly trivial) subgroup of Γ. A point $x \in \Sigma^\Gamma$ is called *k-periodic* if $\mathrm{stab}(x)$ is a subgroup of rank k. When $\Gamma = \mathbb{Z}$, we say that $x \in \Sigma^{\mathbb{Z}}$ is *periodic* if it has a non-trivial stabilizer.

Definition 2.7. A Γ-subshift $X \subseteq \Sigma^\Gamma$ is called *a subshift of finite type (SFT)* if there exists a finite set $W \subset \Gamma$ and a set $\mathcal{F} \subseteq \Sigma^W$ such that

$$X = \left\{x \in \Sigma^\Gamma : \ \forall v \in \Gamma, \ \sigma_v(x)|_W \notin \mathcal{F}\right\}.$$

For every $F \Subset \mathbb{Z}^d$ the space of co-tiles for F is a subshift of finite type, under the natural identification of the space of co-tiles for F with

$$X_F := \left\{x \in \{0,1\}^{\mathbb{Z}^d} : \ \mathbf{1}_F * x = 1\right\}.$$

To see that X_F is indeed an SFT, take $W = -F$ and

$$\mathcal{F} = \left\{p \in \{0,1\}^W \ : \ \sum_{w \in W} p(w) \neq 1\right\},$$

and then

$$X_F = \left\{x \in \{0,1\}^{\mathbb{Z}^d} \ : \ \forall v \in \mathbb{Z}^d, \ \sigma_v(x)|_W \notin \mathcal{F}\right\}.$$

Since a non-empty intersection of SFTs is also an SFT, it follows that the space of joint co-tiles for a collection of tiles is an SFT (unless it is empty).

The following simple result is based on a pigeonhole argument. The proof is well-known and standard, we include it for completeness.

Lemma 2.8. *Every $\mathbb{Z}$-subshift of finite type admits a periodic point.*

Proof. Let $X \subseteq \Sigma^{\mathbb{Z}}$ be a $\mathbb{Z}$-subshift of finite type, where Σ is a finite set. Then by definition, there exists a finite set $W \Subset \mathbb{Z}$ and $\mathcal{F} \subseteq \Sigma^W$ such that

$$X = \left\{x \in \Sigma^{\mathbb{Z}} : \ \forall v \in \mathbb{Z}, \ \sigma_v(x)|_W \notin \mathcal{F}\right\},$$

and $X \neq \emptyset$. Fix $x \in X$, and let $N \in \mathbb{N}$ be an integer bigger than $\max(W) - \min(W)$. Since the set $\Sigma^{\{1,\dots,N\}}$ is finite, by the pigeonhole principle there exist integers $0 \leq i < j \leq |\Sigma|^N$ such that

$$x|_{\{i,\dots,i+N-1\}} = x|_{\{j,\dots,j+N-1\}}.$$

Let $p = j - i$ and define $\hat{x} \in \Sigma^{\mathbb{Z}}$ by

$$\hat{x}_n = x_{i+(n \mod p)}.$$

Then $\hat{x}$ is a periodic point, and for every $n \in \mathbb{Z}$ there exists $t \in \{i, \dots, j-1\}$ such that $\hat{x}|_{W+n} = x|_{W+t}$. Hence, $\hat{x} \in X$, which proves that X admits a periodic point. □

We recall the following result in multidimensional symbolic dynamics.

Lemma 2.9. *Let Γ be a finitely generated abelian group, $\Gamma_0 \leq \Gamma$ a subgroup, and $X \subseteq \Sigma^\Gamma$ a Γ-subshift. Let*

$$X_{\Gamma_0} := \{x \in X \ : \Gamma_0 \leq \operatorname{stab}(x)\}. \tag{6}$$

If $X_{\Gamma_0} \neq \emptyset$ then it is a Γ-subshift. Furthermore, if X is a subshift of finite type then X_{Γ_0} is also a subshift of finite type.

Proof. First, we show that X_{Γ_0} is a subshift. Since Γ is abelian, for every $v \in \Gamma$, $v_0 \in \Gamma_0$ and $y \in X_{\Gamma_0}$ we have

$$\sigma_{v_0}(\sigma_v(y)) = \sigma_v(\sigma_{v_0}(y)) = \sigma_v(y).$$

This shows $\sigma_v(y) \in X_{\Gamma_0}$ for all $v \in \Gamma$ hence X_{Γ_0} is Γ-invariant. To see that X_{Γ_0} is a closed subset of Σ^Γ, consider a sequence $(y_n)_{n\in\mathbb{N}} \in X_{\Gamma_0}$ such that

$$\lim_{n\to\infty} y_n = y \in \Sigma^\Gamma$$

in the product topology. Since each $y_n \in X_{\Gamma_0} \subseteq X$ and X is a closed subset of Σ^Γ, we get $y \in X$. Note that for any $v_0 \in \Gamma_0$,

$$\sigma_{v_0}(y) = \sigma_{v_0}\left(\lim_{n\to\infty} y_n\right) = \lim_{n\to\infty} (\sigma_{v_0}(y_n)) = \lim_{n\to\infty} (y_n) = y,$$

which shows $y \in X_{\Gamma_0}$ and hence X_{Γ_0} is a subshift. Now assuming that X is an SFT we show that X_{Γ_0} is also an SFT. Observe that $X_{\Gamma_0} = X \cap Y$ where

$$Y = \{x \in \Sigma^\Gamma \ : \ \Gamma_0 \leq \operatorname{stab}(x)\}.$$

Since Γ_0 is a subgroup of a finitely generated abelian group it is also finitely generated. Let $\{\gamma_1, \dots, \gamma_r\}$ be a finite generating set for Γ_0. Then

$$Y = \bigcap_{i=1}^{r} \{x \in \Sigma^\Gamma \ : \ \forall v \in \Gamma, \ x_{v+\gamma_i} = x_v\}.$$

To see that Y is an SFT, let $W = \{0, \gamma_1, \dots, \gamma_r\}$ and

$$\mathcal{F} = \left\{w \in \Sigma^W : \ \exists 1 \leq i \leq r \text{ s.t. } w_0 \neq w_{\gamma_i}\right\}.$$

Then

$$Y = \left\{x \in \Sigma^\Gamma \ : \ \forall v \in \Gamma, \ \sigma_v(x)|_W \notin \mathcal{F}\right\}.$$

Hence Y is an SFT, which completes the argument.

□

From Lemma 2.9 we deduce the following:

Lemma 2.10. *Let Γ be a finitely generated abelian group of rank d. If $X \subseteq \Sigma^\Gamma$ is a Γ-subshift of finite type that admits a $(d-1)$-periodic point then it admits a d-periodic point.*

Proof. Suppose $X \subseteq \Sigma^\Gamma$ is a Γ-subshift of finite type that admits a $(d-1)$-periodic point, namely a point $z \in X$ and a subgroup $\Gamma_0 \leq \Gamma$ of rank $d-1$ such that $\mathrm{stab}(z) = \Gamma_0$. Let X_{Γ_0} be given by (6). Then X_{Γ_0} is non-empty, and by Lemma 2.9 it is a subshift of finite type. Because $\mathrm{rank}(\Gamma_0) = d-1$, it follows that $\mathrm{rank}(\Gamma/\Gamma_0) = 1$. Let $v \in \mathbb{Z}^d$ be a vector such that $k \cdot v \notin \Gamma_0$ for all $k \in \mathbb{N}$. Then $\Gamma_0 \oplus \mathbb{Z}v$ is a finite index subgroup of Γ. Let $D \subseteq \Gamma$ be a fundamental domain for $\Gamma_0 \oplus \mathbb{Z}v$, namely a finite set such that $\Gamma_0 \oplus \mathbb{Z}v \oplus D = \Gamma$. Because $D \oplus \mathbb{Z}v$ is a fundamental domain for Γ_0 in Γ, it follows that the restriction map $\rho : X_{\Gamma_0} \to \Sigma^{D\oplus\mathbb{Z}v}$ is injective, where ρ is given by $\rho(x) = x\mid_{D\oplus\mathbb{Z}v}$.

Indeed, the inverse $\rho^{-1} : \rho(X_{\Gamma_0}) \to X_{\Gamma_0}$ is given by $\rho^{-1}(\tilde{x})_u = (\tilde{x})_{u'}$ for $u \in \Gamma$, where u' is is the unique element in $(D \oplus \mathbb{Z}v)$ that satisfies $u - u' \in \Gamma_0$. Using the natural identification $\Sigma^{D\oplus\mathbb{Z}v} \cong (\Sigma^D)^{\mathbb{Z}}$, we can view $\rho(X_{\Gamma_0})$ as a subset of $(\Sigma^D)^{\mathbb{Z}}$, which we denote by $\tilde{X}$.

Let us show that $\tilde{X}$ is a $\mathbb{Z}$-subshift of finite type. Because X is a Γ-subshift of finite type, there exists a finite set $W \subset \Gamma$ and $\mathcal{F} \subset \Sigma^W$ such that X_{Γ_0} is equal to the set of $x \in \Sigma^\Gamma$ satisfying $\sigma_v(x) = x$ and $\sigma_v(x)\mid_W \notin \mathcal{F}$ for all $v \in \Gamma_0$. We can assume without loss of generality that W is a subset of $\mathbb{Z}v \oplus D$, because $\mathbb{Z}v \oplus D$ is a fundmental domain for Γ_0. Let $\tilde{W} = \{n \in \mathbb{Z} : \ (nv + D) \cap W \neq \emptyset\}$. Then $W = \biguplus_{n\tilde{W}} (W \cap (nv + D))$. Thus, there is a natural bijection between Σ^W and $(\Sigma^D)^{\tilde{W}}$. Let $\tilde{\mathcal{F}}$ denote the image of $\mathcal{F}$ under this bijection. Then it follows directly that

$$\tilde{X} = \left\{ x \in (\Sigma^D)^{\mathbb{Z}} : \ \forall v \in \mathbb{Z} : \ \sigma_v(x)\mid_{\tilde{W}} \notin \tilde{\mathcal{F}} \right\}.$$

This proves that $\tilde{X}$ is indeed a $\mathbb{Z}$-subshift of finite type.

Since $\tilde{X}$ is a $\mathbb{Z}$-subshift of finite type, by Lemma 2.8 there exists a periodic point $\tilde{z}$ in $\tilde{X}$. Let $x = \rho^{-1}(\tilde{z})$, then $x \in X$ is a d-periodic point. □

3. The periodic decomposition theorem

The following theorem asserts a certain decomposition for a joint co-tile of k-tuple of tiles in $\mathbb{Z}^d$. The case where $k = 1$ and f is $\{0,1\}$-valued essentially coincides with [GT21a, Theorem 1.7], which is closely related to [Bha20, Theorem 3.3]. In the particular case that the tuple of tiles is independent, Theorem 1.1 is a direct consequence. Namely, the indicator function of any joint co-tile of k independent tiles is a sum of k-periodic functions, each taking values in $[0,1]$. The goal of this section is to prove the periodic decomposition theorem for joint co-tiles and to deduce Theorem 1.1 and Theorem 1.2.

Theorem 3.1 (Periodic decomposition theorem)**.** *Let $F_1, \dots, F_k \Subset \mathbb{Z}^d$, with $0 \in F_i$ for all $1 \leq i \leq k$, and let $f : \mathbb{Z}^d \to \mathbb{Z}$ be a bounded function that satisfies $\mathbf{1}_{F_i} * f = 1$ for all $1 \leq i \leq k$. We denote by $S := |F_1| = \ldots = |F_k|$ (see Proposition 2.5). Then for every $1 \leq i \leq k$ and every $(v_1, \dots, v_i) \in F_1^* \times \ldots \times F_i^*$ there exists a function $\phi_{v_1,\dots,v_i} : \mathbb{Z}^d \to [\min f, \max f]$ with the following properties:*

(a) *For* $i < k$ *we have*

$$\phi_{v_1,\dots,v_i} = 1 - \sum_{v_{i+1}\in F^*_{i+1}} \phi_{v_1,\dots,v_i,v_{i+1}}.$$

(b)

$$f = (-1)^i \sum_{(v_1,\dots,v_i)\in F_1^*\times\dots\times F_i^*} \phi_{v_1,\dots,v_i} + \sum_{j=1}^{i} (-(S-1))^{j-1}.$$

(c) *Let* q *denote the product of all primes less than or equal to* $(\max f - \min f)S$*, then*

$$(\mathbb{Z}qv_1 + \dots + \mathbb{Z}qv_i) \le \operatorname{stab}(\phi_{v_1,\dots,v_i}),$$

(d) $1_{F_j} * \phi_{v_1,\dots,v_i} = 1$ *for all* $1 \le j \le k$*. In particular,* $\phi_{v_1,\dots,v_i}$ *has mean* $1/S$*.*

There are various extensions of Theorem 3.1. Some of these generalizations have further applications. For the sake of readability, we do not state the most general form and instead indicate certain generalizations in the following sections, at the expense of some repetition.

The proof of Theorem 3.1 relies on Lemma 3.2 below. Various versions of this lemma, which is referred to as the dilation lemma, have been proved in [GT21a, Lemma 3.1], [Bha20, Proposition 3.1] for $\Gamma = \mathbb{Z}^d$, $d \ge 1$. We also refer our readers to [Tij95, Theorem 1] where this lemma is proved for integers. The proof is based on some elementary commutative algebra and it easily extends to countable abelian groups. For the sake of self-containment, we include a sketch of the proof below. The proof below is nearly identical to [GT21a, Lemma 3.1], except that we apply the assumption that r is co-prime to the order of torsion elements directly before eq. (7).

Lemma 3.2 (Dilation lemma)**.** *Let* Γ *be a countable abelian group. Let* $0 \in F \Subset \Gamma$, $\ell \in \mathbb{N}$ *and* $f : \Gamma \to \mathbb{Z}$ *a bounded function satisfying*

$$\mathbf{1}_F * f = \ell.$$

Let q_1 *be the product of all primes less than or equal to* $(\max f - \min f)|F|$*, let* q_2 *be the product of all the orders of the torsion elements in* $(F - F)$*, and set* $q = q_1 q_2$*. Then*

$$\mathbf{1}_{rF} * f = \ell,$$

for all $r \in \mathbb{N}$ *such that* $r = 1 \mod q$*.*

Proof. We use the notation $f^{*p} = \underbrace{f * \dots * f}_{\times p}$. For any prime p we have

$$\mathbf{1}_F^{*p} = \left(\sum_{v\in F} \delta_v\right)^{*p} = \sum_{v\in F} \delta_v^{*p} \mod p,$$

where the last equality holds by the Frobenius identity $(f+g)^{*p} = f^{*p} + g^{*p} \mod p$. For integers p that are co-prime to q_2 we have that $p(v_1 - v_2) \neq 0$ for any $v_1 \neq v_2 \in F$, so the function $v \mapsto pv$ is injective on F. Thus:

$$\sum_{v\in F} \delta_v^{*p} = \sum_{v\in F} \delta_{pv} = \mathbf{1}_{pF}. \tag{7}$$

Now convolving both sides of $\mathbf{1}_F * f = \ell$ by $\mathbf{1}_F^{*(p-1)}$ yields $\mathbf{1}_F^{*p} * f = \ell|F|^{p-1}$. Combining the above, for primes p that are co-prime to q_2 we obtain $\mathbf{1}_{pF} * f = \ell|F|^{p-1} \mod p$. If additionally p is co-prime to $|F|$ by Fermat little theorem $|F|^{p-1} = 1 \mod p$, thus

$$\mathbf{1}_{pF} * f = \ell \mod p.$$

Note that both $\mathbf{1}_F * f$ and $\mathbf{1}_{pF} * f$ take values in $[|F| \min f, |F| \max f]$. Recall that $\ell = \mathbf{1}_F * f$, so $\ell \in [|F| \min f, |F| \max f]$. Thus, for p that is also greater than the size of that interval, the above equality holds without the $\mod p$, namely $\mathbf{1}_{pF} * f = \ell$. Finally, for $r = 1 \mod q$, r is a product of primes that satisfy the conditions above, and the result follows by iterating the equation $\mathbf{1}_{pF} * f = \ell$. □

Proof of Theorem 3.1. For $1 \le i \le k$, $(v_1, \dots, v_i) \in F_1^* \times \dots \times F_i^*$ and $N \in \mathbb{N}$ denote:

$$\phi^{(N)}_{v_1,\dots,v_i} := \frac{1}{N^i} \sum_{n_1,\dots,n_i=1}^{N} \delta_{(1+n_1q)v_1+\dots+(1+n_iq)v_i} * f. \tag{8}$$

Let q be the product of all primes less than or equal to $(\max f - \min f)S$. By applying Lemma 3.2 for F_j with $\Gamma = \mathbb{Z}^d$ and $\ell = 1$ we get $\mathbf{1}_{rF_j} * f = 1$ for every $r \in q\mathbb{N} + 1$. Since $0 \in F_j$ we obtain

$$f = 1 - \sum_{v \in F_j^*} \delta_{rv} * f \text{ for every } 1 \le j \le k.$$

For every $N \in \mathbb{N}$, setting $r = 1 + nq$ for $n \in \{1, \dots, N\}$ and taking average we conclude that for every $1 \le j \le k$ we have

$$f = 1 - \sum_{v \in F_j^*} \frac{1}{N} \sum_{n=1}^{N} \delta_{(1+nq)v} * f. \tag{9}$$

Since $\phi^{(N)}_{v_1} = \frac{1}{N} \sum_{n=1}^{N} \delta_{(1+nq)v_1} * f$ this gives (with $j = 1$):

$$f = 1 - \sum_{v_1 \in F_1^*} \phi^{(N)}_{v_1}. \tag{10}$$

For $1 \le i < k$, choose any $(v_1, \dots, v_i) \in F_1^* \times \dots \times F_i^*$ and $1 \le n_1, \dots, n_i \le N$. Setting $j = i + 1$ in (9) and convolving both sides of the equation by $\delta_{(1+n_1q)v_1+\dots+(1+n_iq)v_i}$ we obtain

$$\delta_{(1+n_1q)v_1+\dots+(1+n_iq)v_i} * f = 1 - \sum_{v_{i+1} \in F_{i+1}^*} \frac{1}{N} \sum_{n_{i+1}=1}^{N} \delta_{(1+n_1q)v_1+\dots+(1+n_iq)v_i+(1+n_{i+1}q)v_{i+1}} * f.$$

By averaging over $1 \le n_1, \dots, n_i \le N$ and applying the definition in (8) we obtain that

$$\phi^{(N)}_{v_1,\dots,v_i} = 1 - \sum_{v_{i+1} \in F_{i+1}^*} \frac{1}{N^{i+1}} \sum_{n_1,\dots,n_{i+1}=1}^{N} \delta_{(1+n_1q)v_1+\dots+(1+n_{i+1}q)v_{i+1}} * f = 1 - \sum_{v_{i+1} \in F_{i+1}^*} \phi^{(N)}_{v_1,\dots,v_{i+1}}. \tag{11}$$

Since $|F_i^*| = S - 1$ for $1 \le i \le k$, using (10), (11) and an inductive argument we obtain that for every $N \in \mathbb{N}$ and $1 \le i \le k$ we have

$$f = \sum_{j=1}^{i} (-(S-1))^{j-1} + (-1)^i \sum_{(v_1,\dots,v_i) \in F_1^* \times \dots \times F_i^*} \phi^{(N)}_{v_1,\dots,v_i} \tag{12}$$

Notice that the functions $\delta_{(1+n_1q)v_1+\ldots+(1+n_iq)v_i} * f$ are bounded between $\min f$ and $\max f$, thus by (8), the functions $\phi^{(N)}_{v_1,\ldots,v_i}$ are bounded between $\min f$ and $\max f$ for every $(v_1,\ldots,v_i) \in F_1^* \times \ldots \times F_i^*$. In particular, for every $(v_1,\ldots,v_i) \in F_1^* \times \ldots \times F_i^*$ the sequence of functions $(\phi^N_{v_1,\ldots,v_i})_{N\in\mathbb{N}}$ is uniformly bounded, hence by ArzelAscoli theorem (or by a Cantor diagonalization argument), it converges along a subsequence. We denote the limit by $\phi_{v_1,\ldots,v_i}$. Then for every $(v_1,\ldots,v_i) \in F_1^* \times \ldots \times F_i^*$ we have

$$\min f \le \phi_{v_1,\ldots,v_i} \le \max f,$$

and in view of (11) and (12) we have achieved (a) and (b).

To see (c), using (8), a standard telescoping argument shows that for every $w \in \mathbb{Z}^d$, $v = (v_1,\ldots,v_i) \in F_1^* \times \ldots \times F_i^*$ and every $1 \le j \le i$ we have

$$\left|\phi^{(N)}_{v_1,\ldots,v_i}(w+qv_j) - \phi^{(N)}_{v_1,\ldots,v_i}(w)\right| \le \frac{2N^{k-1}}{N^k} = \frac{2}{N}.$$

Thus for every $(v_1,\ldots,v_i) \in F_1^* \times \ldots \times F_i^*$ the function $\phi_{v_1,\ldots,v_i}$ is qv_j-periodic for every $1 \le j \le i$. It is left to see (d). Clearly, since $\mathbf{1}_{F_j} * f = 1$, for every $1 \le i,j \le k$, $(v_1,\ldots,v_i) \in F_1^* \times \ldots \times F_i^*$ and $n_1,\ldots,n_i \in \mathbb{N}$ we have $\mathbf{1}_{F_j} * (\delta_{(1+n_1q)v_1+\ldots(1+n_iq)v_i} * f) = 1$. Thus, by (8), $\mathbf{1}_{F_j} * \phi^{(N)}_{v_1,\ldots,v_i} = 1$ for every $N \in \mathbb{N}$ and therefore $\mathbf{1}_{F_j} * \phi_{v_1,\ldots,v_i} = 1$ for every $1 \le i,j \le k$. In particular, by Proposition 2.5, $\phi_{v_1,\ldots,v_i}$ has mean $1/S$. ☐

Remark 3.3. Under the assumption that f is $\{0,1\}$-valued, it directly follows from Theorem 3.1, part (a), that for every $1 \le i < k$ and every $(v_1,\ldots,v_i) \in F_1^* \times \ldots \times F_i^*$, the sum $\sum_{v_{i+1}\in F^*_{i+1}} \phi_{v_1,\ldots,v_i,v_{i+1}}$ is a $[0,1]$-valued function. Theorem 3.1, where $k = 1$ and f is $\{0,1\}$-valued, coincides with [GT21a, Theorem 1.7]. We will not make use of the property that $\mathbf{1}_{F_j} * \phi_{v_1,\ldots,v_i} = 1$ in this paper. We mention it only for completeness and possibly for future reference. The fact that the functions ϕ_{v_1} each have mean $1/S$ played an implicit role in [Bha20].

Using the assumption that the tuple of tiles is independent Theorem 1.1 is an immediate corollary of Theorem 3.1, with f being a $\{0,1\}$-valued function. The proof of Theorem 1.2 is straightforward.

Proof of Theorem 1.2. Suppose that $(F_1,\ldots,F_d)$ is an independent tuple of tiles in $\mathbb{Z}^d$ and that $f : \mathbb{Z}^d \to \mathbb{Z}$ is a bounded function satisfying $\mathbf{1}_{F_i} * f = 1$ for all $1 \le i \le d$. By Proposition 2.5, we have $|F_1| = \ldots = |F_d| := S$. Let q be the product of all primes less than or equal to $(\max f - \min f)S$ and let

$$L = \bigcap_{(v_1,\ldots,v_d)\in F_1^*\times\ldots\times F_d^*} q\mathbb{Z}v_1 + \ldots + q\mathbb{Z}v_d.$$

Apply Theorem 3.1 with $k = d$. It follows that f is a sum of functions whose stabilizers are rank d-subgroups, more precisely,

$$f = (-1)^d \sum_{(v_1,\ldots,v_d)\in F_1^*\times\ldots\times F_d^*} \phi_{v_1,\ldots,v_d} + \sum_{j=1}^{d} (-(S-1))^{j-1},$$

and for each $(v_1,\ldots,v_d) \in F_1^* \times \ldots \times F_d^*$ we have that $q\mathbb{Z}v_1 + \ldots + q\mathbb{Z}v_d \le \mathrm{stab}(\phi_{v_1,\ldots,v_d})$. By the above, $\mathrm{stab}(f)$ contains the intersection of $\mathrm{stab}(\phi_{v_1,\ldots,v_d})$ over $(v_1,\ldots,v_d) \in F_1^* \times \ldots \times F_d^*$, that in turn contains L.

By the assumption that the tuple $(F_1,\dots,F_d)$ is independent, $q\mathbb{Z}v_1+\dots+q\mathbb{Z}v_d$ is a finite index subgroup of $\mathbb{Z}^d$ for every $(v_1,\dots,v_d)\in F_1^*\times\dots\times F_d^*$. Since L is an intersection of finitely many finite index subgroups, L is also a finite index subgroup. This proves that f is periodic. □

4. Joint co-tilings in finitely generated abelian groups

It is natural to ask which of the results about tilings generalize from $\mathbb{Z}^d$ to more general groups. An inspection of the proof of Theorem 3.1 reveals that the statement still holds, and the same proof applies, if we replace $\mathbb{Z}^d$ by an arbitrary countable abelian group Γ, and change the value of q in Theorem 3.1 (c) by multiplying it with the product of the orders of all torsion elements in $F-F$.

There is a simple observation that allows one to reduce statements about tilings of countable abelian groups by a finite set to the finitely generated case: Let Γ be a countable abelian group and let $F\Subset\Gamma$ with $0\in F$. Let Γ_0 denote the group generated by the difference set $F-F$. The assumption $0\in F$ implies that $F\Subset\Gamma_0$. Then for any co-tile A of F we have that $A\cap\Gamma_0$ is a co-tile of F in Γ_0, and tilings of Γ by F decompose into tilings of cosets of Γ_0 in Γ. A corresponding statement is true also for a tuple of tiles $(F_1,\dots,F_k)$ and a joint co-tile.

Recall that $g_1,\dots,g_k$ in a countable abelian group Γ are called *independent* if the equation $\sum_{j=1}^k n_jg_j=0$, with $n_1,\dots,n_k\in\mathbb{Z}$, implies that $n_1=\dots=n_k=0$. With this definition, Theorem 1.1 extends directly as follows:

Theorem 4.1. *Let Γ be a countable abelian group. For every $k\in\mathbb{N}$ the indicator function of any joint co-tile for k independent tiles in Γ is equal, up to a constant, to a sum of $[0,1]$-valued functions whose stabilizer has rank at least k.*

Similarly, Theorem 1.2 extends as follows:

Theorem 4.2. *Let Γ be a finitely generated abelian group of rank d. Any joint co-tile for d independent tiles in Γ has a finite orbit.*

A quick remark about the condition of independence for a tuple of tiles for finitely generated abelian groups with non-trivial torsion: If Γ is of the form $\Gamma=\mathbb{Z}^d\times G$ where G is a finite abelian group and $(F_1,\dots,F_k)$ is an independent tuple of tiles in Γ, then the only torsion element in each of the sets F_i is 0. For this reason, Newman's theorem (i.e. any tiling of $\mathbb{Z}$ by a finite set is periodic) does not hold in abelian groups Γ that are finite extensions of $\mathbb{Z}$. Indeed, take $\Gamma=\mathbb{Z}\times G$, where G is a finite abelian group. Take $F=\{1\}\times G\Subset\Gamma$, then the co-tiles of F are all the sets $A\subset\Gamma$ of the following form:

$$A=\{(n,g_n):\ n\in\mathbb{Z}\},$$

for some sequence $(g_n)_{n\in\mathbb{Z}}$ of elements in G. In particular, it is no longer true that any co-tile of F must be periodic, unless G is trivial. Nonetheless, if G is a finite cyclic group of prime order, then the only obstructions to extending Newman's theorem are of this form.

Proposition 4.3. *If $\Gamma=\mathbb{Z}\times(\mathbb{Z}/p\mathbb{Z})$ for some prime number p and $F\Subset\Gamma$ is a finite set, then every co-tile of F is periodic, unless F is of the form $F=\tilde{F}\times(\mathbb{Z}/p\mathbb{Z})$ for some finite tile $\tilde{F}\Subset\mathbb{Z}$, in which case the co-tiles of F are all of the form*

$$A=\{(n,g_n):n\in\tilde{A}\},\ g_n\in\mathbb{Z}/p\mathbb{Z},\tag{13}$$

where $\tilde{A}$ is a co-tile of $\tilde{F}\Subset\mathbb{Z}$, which by Newman's theorem must be periodic.

The proof of the proposition relies on the following generalization of Theorem 3.1.

Theorem 4.4. *Let Γ be a countable abelian group, $F_1, \ldots, F_k \Subset \Gamma$ such that $|F_i| = S$, and $0 \in F_i$ for all $1 \le i \le k$, and let $f : \Gamma \to \mathbb{Z}$ be a bounded function that satisfies $\mathbf{1}_{F_i} * f = 1$ for all $1 \le i \le k$. For every $1 \le i \le k$, let F_i^{Tor} denote the intersection of F_i with the torsion subgroup of Γ, and let $F_i^* = F_i \setminus F_i^{\mathrm{Tor}}$. Then for every $1 \le i \le k$ and every $(v_1, \ldots, v_i) \in F_1^* \times \ldots \times F_i^*$ there exists a function $\phi_{v_1,\ldots,v_i} : \Gamma \to [\min f, \max f]$ with the following properties:*

(a) *For $i < k$ we have*

$$\mathbf{1}_{F_{i+1}^{\mathrm{Tor}}} * \phi_{v_1,\ldots,v_i} = 1 - \sum_{v_{i+1} \in F_{i+1}^*} \phi_{v_1,\ldots,v_i,v_{i+1}}.$$

(b) *For every $1 \le i \le k$ there is an integer constant C_i such that*

$$\mathbf{1}_{F_1^{\mathrm{Tor}}} * \ldots * \mathbf{1}_{F_i^{\mathrm{Tor}}} * f = (-1)^i \sum_{(v_1,\ldots,v_i) \in F_1^* \times \ldots \times F_i^*} \phi_{v_1,\ldots,v_i} + C_i.$$

(c) *Let q_1 be the product of all primes less than or equal to $(\max f - \min f)S$, let q_2 be the product of all the orders of the torsion elements in the sets $F_i - F_i$, for $1 \le i \le k$, and set $q = q_1 q_2$. Then*

$$(\mathbb{Z}qv_1 + \ldots + \mathbb{Z}qv_i) \le \mathrm{stab}(\phi_{v_1,\ldots,v_i}),$$

(d) *$1_{F_j} * \phi_{v_1,\ldots,v_i} = 1$ for all $1 \le j \le k$. In particular, $\phi_{v_1,\ldots,v_i}$ has mean $1/S$.*

The proof of Theorem 4.4 below is a minor adaptation of the proof of Theorem 3.1. Note that in the case where Γ is a torsion free abelian group, $F_i^{\mathrm{Tor}} = \{0\}$. In particular, when $\Gamma = \mathbb{Z}^d$, Theorem 4.4 coincides with Theorem 3.1.

Proof. By applying Lemma 3.2 for F_i with $\ell = 1$ and q as in (c) we get $\mathbf{1}_{rF_i} * f = 1$ for every $r \in q\mathbb{N} + 1$. Because $r = 1 \mod q$, we have $rF_i^{\mathrm{Tor}} = F_i^{\mathrm{Tor}}$. Since $F_i = F_i^{\mathrm{Tor}} \uplus F_i^*$ we have

$$\mathbf{1}_{F_i^{\mathrm{Tor}}} * f = 1 - \sum_{v \in F_i^*} \delta_{rv} * f \text{ for every } 1 \le i \le k.$$

For every $N \in \mathbb{N}$, setting $r = 1 + nq$ for $n \in \{1, \ldots, N\}$ and taking average we conclude that for every $1 \le j \le k$ we have

$$\mathbf{1}_{F_j^{\mathrm{Tor}}} * f = 1 - \sum_{v_j \in F_j^*} \frac{1}{N} \sum_{n_j=1}^{N} \delta_{(1+n_j q)v_j} * f. \tag{14}$$

Applying (14) with $j = i+1$, convolving both sides by $\delta_{(1+n_1q)v_1+\ldots+(1+n_iq)v_i}$ and taking average over $\frac{1}{N^i}\sum_{n_1,\ldots,n_i=1}^N$ yields

$$\mathbf{1}_{F_{i+1}^{\mathrm{Tor}}} * \left[\frac{1}{N^i} \sum_{n_1,\ldots,n_i=1}^{N} \delta_{(1+n_1q)v_1+\ldots+(1+n_iq)v_i} * f \right] =$$

$$1 - \sum_{v_{i+1} \in F_{i+1}^*} \frac{1}{N^{i+1}} \sum_{n_1,\ldots,n_i,n_{i+1}=1}^{N} \delta_{(1+n_1q)v_1+\ldots+(1+n_iq)v_i+(1+n_{i+1}q)v_{i+1}} * f.$$

Defining $\phi^{(N)}_{v_1,\dots,v_i} = \frac{1}{N^i}\sum_{n_1,\dots,n_i=1}^{N} \delta_{(1+n_1q)v_1+\dots+(1+n_iq)v_i} * f$, as in (8), we obtain

$$\mathbf{1}_{F^{\mathrm{Tor}}_{i+1}} * \phi^{(N)}_{v_1,\dots,v_i} = 1 - \sum_{v_{i+1}\in F^*_{i+1}} \phi^{(N)}_{v_1,\dots,v_i,v_{i+1}}. \tag{15}$$

Note that (14) with $j = 1$ becomes $\mathbf{1}_{F^{\mathrm{Tor}}_1} * f = 1 - \sum_{v_1\in F^*_1} \phi^{(N)}_{v_1}$. Convolving both sides by $\mathbf{1}_{F^{\mathrm{Tor}}_2}$ and using (15) with $i = 1$ gives

$$\mathbf{1}_{F^{\mathrm{Tor}}_1} * \mathbf{1}_{F^{\mathrm{Tor}}_2} * f = |F^{\mathrm{Tor}}_2| - \sum_{v_1\in F^*_1} \mathbf{1}_{F^{\mathrm{Tor}}_2} * \phi^{(N)}_{v_1} = |F^{\mathrm{Tor}}_2| - \sum_{v_1\in F^*_1}\left(1 - \sum_{v_2\in F^*_2} \phi^{(N)}_{v_1,v_2}\right).$$

By an inductive argument we obtain that for every $N \in \mathbb{N}$ and $1 \le i \le k$ there is a constant $C_i \in \mathbb{Z}$, that does not depend on N, such that

$$\mathbf{1}_{F^{\mathrm{Tor}}_1} * \dots \mathbf{1}_{F^{\mathrm{Tor}}_i} * f = C_i + (-1)^i \sum_{(v_1,\dots,v_i)\in F^*_1\times\dots\times F^*_i} \phi^{(N)}_{v_1,\dots,v_i}. \tag{16}$$

Items (a) and (b) follow from (15) and (16) respectively. The rest of the proof is completely identical to the proof of Theorem 3.1 and therefore omitted. □

Lemma 4.5. *Let p be a prime number and let $\emptyset \ne F_0 \subsetneq \mathbb{Z}/p\mathbb{Z}$. Then $\mathbf{1}_{F_0}$ is an invertible element of the ring $\mathbb{Q}^{\mathbb{Z}/p\mathbb{Z}}$, where multiplication in the ring is convolution. In other words, there exists $g \in \mathbb{Q}^{\mathbb{Z}/p\mathbb{Z}}$ such that $g * \mathbf{1}_{F_0} = \delta_0$.*

Proof. Consider the ring $\mathbb{Q}[x]/\langle x^p - 1\rangle$ (with operations of addition and multiplication of polynomials). It is easy to check that this ring is isomorphic as a ring to $\mathbb{Q}^{\mathbb{Z}/p\mathbb{Z}}$, with the operations of pointwise addition and convolution. The isomorphism is given by identifying an element

$$\sum_{i=0}^{p-1} a_i x^i + \langle x^p - 1\rangle \in \mathbb{Q}[x]/\langle x^p - 1\rangle$$

with the function $f \in \mathbb{Q}^{\mathbb{Z}/p\mathbb{Z}}$ given by $f(i + p\mathbb{Z}) = a_i$.

Let $F_0 \subset \mathbb{Z}/p\mathbb{Z}$ be a non-empty proper subset of $\mathbb{Z}/p\mathbb{Z}$. Then $\mathbf{1}_{F_0} \in \mathbb{Q}^{\mathbb{Z}/p\mathbb{Z}}$ is naturally identified with the coset of the polynomial $P(x) = \sum_{(i+p\mathbb{Z})\in F_0} x^i$ in $\mathbb{Q}[x]/\langle x^p - 1\rangle$. Then the assumption that F_0 is a non-empty proper subset of $\mathbb{Z}/p\mathbb{Z}$ implies that the polynomial P is co-prime to the cyclotomic polynomial of order p, $\Phi_p = \sum_{i=0}^{p-1} x^i$. Since $P(1) = |F_0| \ne 0$ it follows that P is co-prime to $x - 1$. Because $x^p - 1 = \Phi_p(x)(x - 1)$, it follows that P is co-prime to $x^p - 1$. Hence there exists polynomials $Q_1, Q_2 \in \mathbb{Q}[x]$ such that

$$1 = Q_1(x)P(x) + Q_2(x)(x^p - 1).$$

This means that in the ring $\mathbb{Q}[x]/\langle x^p - 1\rangle$, the coset of $Q_1(x)P(x)$ is the same as the coset of the polynomial 1. Since the coset of the polynomial 1 in $\mathbb{Q}[x]/\langle x^p - 1\rangle$ corresponds to $\delta_0 \in \mathbb{Q}^{\mathbb{Z}/p\mathbb{Z}}$, this implies that $g * \mathbf{1}_{F_0} = \delta_0$, where $g \in \mathbb{Q}^{\mathbb{Z}/p\mathbb{Z}}$ is the element corresponding to the coset of Q_1. □

Proof of Proposition 4.3. Let p be a prime number and $F \Subset \mathbb{Z} \times (\mathbb{Z}/p\mathbb{Z})$ be a finite set. Suppose $A \subset \mathbb{Z} \times (\mathbb{Z}/p\mathbb{Z})$ satisfies $\mathbf{1}_F * \mathbf{1}_A = 1$. Applying Theorem 4.4 with $\Gamma = \mathbb{Z} \times (\mathbb{Z}/p\mathbb{Z})$ $k = 1$, $F_1 = F$ and $f = \mathbf{1}_A$, we conclude that $\mathbf{1}_{F^{\mathrm{Tor}}} * \mathbf{1}_A$ is a sum functions having infinite stabilizer, hence $\mathbf{1}_{F^{\mathrm{Tor}}} * \mathbf{1}_A$ is periodic.

First, assume that there is a set $\tilde{F} \Subset \mathbb{Z}$ such that $F = \tilde{F} \times \mathbb{Z}/p\mathbb{Z}$. So $\mathbf{1}_F = \mathbf{1}_{\tilde{F}\times\{0\}} * \mathbf{1}_{\{0\}\times(\mathbb{Z}/p\mathbb{Z})}$. Thus $\mathbf{1}_{\tilde{F}\times\{0\}} * \mathbf{1}_{\{0\}\times(\mathbb{Z}/p\mathbb{Z})} * \mathbf{1}_A = 1$. This implies that $\mathbf{1}_{\{0\}\times(\mathbb{Z}\times p\mathbb{Z})} * 1_A \leq 1$, so for every $n \in \mathbb{Z}$ there exists at most one element $g_n \in \mathbb{Z}/p\mathbb{Z}$ such that $(n, g_n) \in A$. Hence, in this case, A is of the form (13) for some set $\tilde{A} \subset \mathbb{Z}$. It follows that $\mathbf{1}_{\tilde{F}} * \mathbf{1}_{\tilde{A}} = 1$, where the convolution here is with respect to the group $\mathbb{Z}$.

Now suppose that F is not of the above form. This means that there exists $n \in \mathbb{Z}$ such that $F \cap (\{n\} \times \mathbb{Z}/p\mathbb{Z})$ is a non-empty proper subset of $\{n\} \times (\mathbb{Z}/p\mathbb{Z})$. By translating F we can assume without loss of generality that F^{Tor} is neither empty nor equal to $\{0\} \times (\mathbb{Z}/p\mathbb{Z})$. Then there exists a non-empty proper subset $F_0 \subset \mathbb{Z}/p\mathbb{Z}$ such that $F^{\mathrm{Tor}} = \{0\} \times F_0$. In this case, by Lemma 4.5, there exists $g : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Q}$ such that $g * \mathbf{1}_{F_0} = \delta_0$, where the convolution is in $(\mathbb{Z}/p\mathbb{Z})$. Let $\tilde{g} : \mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \to \mathbb{Q}$ be given by $\tilde{g}(0, i) = g(i)$ for $i \in \mathbb{Z}/p\mathbb{Z}$ and $g(n, i) = 0$ for every $n \in \mathbb{Z} \setminus \{0\}$ and $i \in \mathbb{Z}/p\mathbb{Z}$. Then $\tilde{g} * \mathbf{1}_{F^{\mathrm{Tor}}} = \delta_0$, where this time the convolution is in $\mathbb{Z} \times (\mathbb{Z}/p\mathbb{Z})$. Since $\mathbf{1}_{F^{\mathrm{Tor}}} * \mathbf{1}_A$ is periodic, so is $\tilde{g} * \mathbf{1}_{F^{\mathrm{Tor}}} * \mathbf{1}_A = \mathbf{1}_A$.

We have thus shown that in the case that F is not of the form $F = \tilde{F} \times (\mathbb{Z}/p\mathbb{Z})$ for some $\tilde{F} \Subset \mathbb{Z}$, every co-tile is periodic. □

5. Property ($\star$) implies $(d-1)$-piecewise periodicity

In this section, we use property ($\star$) to deduce Theorem 1.5. To this end, we will use Theorem 2.4, which is a version of Weyl's equidistribution theorem for polynomials in several variables. The relevance of Weyl's equidistribution theorem to our setting comes from Lemma 5.1 below. We note that similar arguments have appeared earlier in [Bha20], [KS20] and [GT21a].

Lemma 5.1. *Suppose $g, g_1, \ldots, g_m : \Gamma_1 \to \Gamma_2$ are functions, where Γ_1, Γ_2 are abelian groups, such that $\sum_{i=1}^m g_i = g$. Suppose g is a polynomial of degree at most $r \in \mathbb{N}$ with respect to a subgroup $\Gamma_0 \leq \Gamma_1$. For any $1 \leq i < j \leq m$ define the group $L_{i,j} = \mathrm{stab}(g_i) + \mathrm{stab}(g_j)$, and let $L = \bigcap_{1\leq i<j\leq m} L_{i,j} \cap \Gamma_0$. Then each g_i is a polynomial of degree at most $\max\{m-1, r\}$ with respect to L. In particular, if Γ_0 and $L_{i,j}$ has finite index in Γ_1 for every $1 \leq i < j \leq m$, then L has finite index in Γ_1, and each g_i is a polynomial with respect to a finite index subgroup of Γ_1.*

Proof. We prove the claim by induction on m. If $m = 1$ then $g_1 = g$, so the claim holds. For $m > 1$, take $v \in L$, then in particular $v \in L_{1,2} \cap \Gamma_0$ and thus $v = v_1 + v_2$ for some $v_1 \in \mathrm{stab}(g_1)$ and $v_2 \in \mathrm{stab}(g_2)$. Note that for every function $f : \Gamma_1 \to \Gamma_2$, the identity $D_v f = D_{v_1} f \circ \sigma_{v_2} + D_{v_2} f$ holds, where $\sigma_u : \Gamma_1 \to \Gamma_1$ denotes the shift by u, $\sigma_u(w) = w - u$. Since $D_{v_1} g_1 = 0$, applying this identity to $g_1 = -\sum_{i=2}^m g_i + g$ yields

$$D_v g_1 = D_{v_2} g_1 = -D_{v_2}\left(\sum_{i=2}^m g_i - g\right).$$

Since $D_{v_2} g_2 = 0$ we have

$$D_v g_1 + \sum_{i=3}^m D_{v_2} g_i = D_{v_2} g. \tag{17}$$

Note that $v_2 \in \Gamma_0$, hence $D_{v_2} g$ is a polynomial of degree at most $r - 1$ with respect to Γ_0. So by the induction hypothesis, each summand on the left-hand side in (17) is a polynomial of degree at most $\max\{m-2, r-1\}$ with respect to a subgroup L', defined in a similar way to L using the functions $D_v g_1, D_{v_2} g_3, \ldots, D_{v_2} g_m$. In particular, for every $v \in L$ the function $D_v g_1$ is a polynomial of degree at most $\max\{m-2, r-1\}$ with respect to L'.

Now observe that for every $f : \Gamma_1 \to \Gamma_2$ and $v \in \Gamma_1$ we have $stab(f) \subseteq stab(D_v f)$, thus $L \le L'$ and for every $v \in L$ we, in particular, have that $D_v g_1$ is a polynomial of degree at most $\max\{m-2, r-1\}$ with respect to L. In a similar way for $2 \le i \le m$ and every $v \in L$, each $D_v g_i$ is a polynomial of degree at most $\max\{m-2, r-1\}$ with respect to L, which completes the proof.

□

Lemma 5.2. *Suppose $g : \mathbb{Z}^d \to [0,1]$ is a function such that:*

(1) $g \mod 1$ is a polynomial with respect to a finite index subgroup of $\mathbb{Z}^d$.
(2) g is a sum of finitely many non-negative $(d-1)$-periodic functions.

Then there exists a finite index subgroup $\Gamma \le \mathbb{Z}^d$ such that the restriction of g to each coset of Γ is $(d-1)$-periodic.

Proof. Suppose $g = \sum_{i=1}^m g_i$, where $g_i : \mathbb{Z}^d \to [0,1]$ and $\operatorname{rank}(\operatorname{stab}(g_i)) \ge d-1$. In case that $\operatorname{rank}\left(\bigcap_{i=1}^m \operatorname{stab}(g_i)\right) \ge d-1$, the function g is $(d-1)$-periodic and the assertion follows. Otherwise, by summing together some of the g_i's we can assume without loss of generality that $\operatorname{stab}(g_i) + \operatorname{stab}(g_j)$ is a finite index subgroup of $\mathbb{Z}^d$, for every $i \neq j$. By Lemma 5.1, because g modulo 1 is a polynomial with respect to a finite index subgroup, we conclude that each of the g_i's modulo 1 are polynomials with respect to a finite index subgroup $\Gamma_0 \le \mathbb{Z}^d$. Let

$$\Gamma = \Gamma_0 \cap \bigcap_{i \neq j} \left(\operatorname{stab}(g_i) + \operatorname{stab}(g_j)\right).$$

We will show that g is $(d-1)$-periodic on each coset of Γ. Since $\Gamma \le \Gamma_0$, each g_i modulo 1 is also a polynomials with respect to Γ. Hence by Weyl's equidistribution theorem (Theorem 2.4), every g_i modulo 1 is either equidistributed or periodic, on each coset of Γ.

Fix $u \in \mathbb{Z}^d$. Let $g^{(u)} : (u + \Gamma) \to [0,1]$ denote the restriction of g to this coset. We consider 3 cases:

(1) Suppose there exists $1 \le i \le m$ and $v \in (u + \Gamma)$ such that $g_i(v) = 1$. Then because $0 \le g(v) \le 1$ and $g_j(v) \ge 0$, we conclude that $g_j(v) = 0$ for all $j \neq i$. But $g_i(v) = 1$ implies that $g_i(v + w_1) = 1$ for all $w_1 \in \operatorname{stab}(g_i)$ so by the same argument $g_j(v + w_1) = 0$ for all $w_1 \in \operatorname{stab}(g_i)$. Thus, $g_j(v + w_1 + w_2) = 0$ for all $w_1 \operatorname{stab}(g_i)$ and $w_2 \in \operatorname{stab}(g_j)$. Since $\Gamma \le \operatorname{stab}(g_i) + \operatorname{stab}(g_j)$, we conclude that g_j is zero on the coset $u + \Gamma$, for all $j \neq i$. This shows that in this case $g^{(u)} = g_i$ on $u + \Gamma$, and in particular $g^{(u)}$ is $(d-1)$-periodic. So in the remaining cases we can assume that none of the g_i's are equal to one, hence the g_i's obtain values in the interval $[0,1)$.

(2) Suppose there exists $1 \le i \le m$ such that g_i is equidistributed modulo 1 on $u + \Gamma$. Let $0 < \epsilon < 1$ be smaller than all the non-zero values obtained by the (possibly empty) set of g_j that are periodic modulo 1. Because g_i is equidistributed modulo 1 on $u + \Gamma$, there exists $v \in u + \Gamma$ such that $g_i(v) > 1 - \epsilon$. Thus, $g_j(v) < \epsilon$ for all $j \neq i$. As in the previous part, using $\Gamma \le \operatorname{stab}(g_i) + \operatorname{stab}(g_j)$, we conclude that $g_j(w) < \epsilon$ for all $j \neq i$ and all $w \in u + \Gamma$. This tells us that in particular that g_j is not equidistributed modulo 1 on $u + \Gamma$. By the choice of ϵ, $g_j(w) = 0$ for every periodic $j \neq i$ and every $w \in u + \Gamma$. We conclude also in this case that $g = g_i$ on $u + \Gamma$ and particular $g^{(u)}$ is $(d-1)$-periodic.

(3) The remaining case is that all the g_i's modulo 1 are periodic on $u+\Gamma$, but since they take values in $[0,1)$, the g_i's themselves are all d-periodic. It follows in this case that g^u is d-periodic, as the sum of d-periodic functions (and in particular $(d-1)$-periodic).

□

Proof of Theorem 1.5. We conveniently assume $d>2$, because the case $d=2$ is covered by [GT21a]. Suppose that $A\subset\mathbb{Z}^d$ satisfies $F_i\oplus A=\mathbb{Z}^d$ for all $1\le i\le d-1$, where $(F_1,\dots,F_{d-1})$ is a tuple of tiles in $\mathbb{Z}^d$ that has property $(\star)$, see Definition 1.4. Let $\phi_{v_1,\dots,v_{d-1}}:\mathbb{Z}^d\to[0,1]$ be as in Theorem 3.1, applied for $k=d-1$ and $f=\mathbf{1}_A$. Given $(v_1,\dots,v_{d-2})\in F_1^*\times\dots\times F_{d-2}^*$ and a $(d-1)$-dimensional subspace $V<\mathbb{R}^d$ such that $v_1,\dots,v_{d-2}\in V$, define

$$\psi_V=\sum_{w_{d-1}\in F_{d-1}^*\cap V}\phi_{v_1,\dots,v_{d-2},w_{d-1}}.$$

Note that by the independence of $(F_1,\dots,F_{d-1})$, every $(d-1)$-tuple in $F_1^*\times\dots\times F_{d-1}^*$ spans a $(d-1)$-dimensional subspace. Denote by H the set (counted without multiplicity) of all $(d-1)$-dimensional subspaces of $\mathbb{R}^d$ spanned by $(d-1)$-tuples in $F_1^*\times\dots\times F_{d-1}^*$, and for $(v_1,\dots,v_{d-2})\in F_1^*\times\dots\times F_{d-2}^*$ let $H(v_1,\dots,v_{d-2})\subset H$ be the set of such subspaces of dimension $(d-1)$ that contain $v_1,\dots,v_{d-2}$. Thus, for every fixed tuple $(v_1,\dots,v_{d-2})\in F_1^*\times\dots\times F_{d-2}^*$ we have

$$\sum_{w_{d-1}\in F_{d-1}^*}\phi_{v_1,\dots,v_{d-2},w_{d-1}}=\sum_{V\in H(v_1,\dots,v_{d-2})}\psi_V. \tag{18}$$

By property $(\star)$, $\{H(v_1,\dots,v_{d-2})\ :\ (v_1,\dots,v_{d-2})\in F_1^*\times\dots\times F_{d-2}^*\}$ is a partition of H, therefore

$$\sum_{(v_1,\dots,v_{d-1})\in F_1^*\times\dots\times F_{d-1}^*}\phi_{v_1,\dots,v_{d-1}}=\sum_{V\in H}\psi_V. \tag{19}$$

It follows that the functions ψ_V possess the following three properties:

(i)
$$1-\phi_{v_1,\dots,v_{d-2}}=\sum_{V\in H(v_1,\dots,v_{d-2})}\psi_V.$$

(ii) $\operatorname{stab}(\psi_V)$ is a rank $(d-1)$ subgroup of $V\cap\mathbb{Z}^d$.

(iii) ψ_V modulo 1 is a polynomial with respect to a finite index subgroup of $\mathbb{Z}^d$.

Indeed, property (i) is a direct consequence of Theorem 3.1 part (a) with $i=d-1$, combined with (18). Property (ii) follows from Theorem 3.1 part (c). Setting $\widetilde{\psi_V}=\psi_V \mod 1$, the equation in Theorem 3.1 part (b) (with $f=\mathbf{1}_A$ and $i=d-1$), combined with (19), yields that $\sum_{V\in H}\widetilde{\psi_V}=0$. By property (ii), $\operatorname{stab}(\widetilde{\psi_V})+\operatorname{stab}(\widetilde{\psi_{V'}})$ is a finite index subgroup of $\mathbb{Z}^d$ whenever $V,V'\in H$ and $V\neq V'$. Thus property (iii) follows from Lemma 5.1.

In view of these three properties, Lemma 5.2 can be applied to $g=1-\phi_{v_1,\dots,v_{d-2}}$, for any $(v_1,\dots,v_{d-2})\in F_1^*\times\dots\times F_{d-2}^*$. This implies that there is a finite index subgroup $\Gamma_{d-2}\le\mathbb{Z}^d$ such that each $\phi_{v_1,\dots,v_{d-2}}$ is a polynomial with respect to Γ_{d-2}, and its restriction to every coset $u+\Gamma_{d-2}$ is $(d-1)$-periodic.

Next, we iterate the above argument using the recursion formula in part (a) of Theorem 3.1 combined with Lemma 5.2. In turn, this yields a finite index subgroup $\Gamma_1\le\mathbb{Z}^d$ such that each ϕ_{v_1} is a polynomial with respect to Γ_1, and its restriction to every coset $u+\Gamma_1$ is $(d-1)$-periodic. By part (b) of Theorem 3.1 with $i=1$ we have that

$$1-\mathbf{1}_A = \sum_{v_1\in F_1^*} \phi_{v_1}.$$

So applying Lemma 5.2 to $g = 1-\mathbf{1}_A$, we obtain a finite index subgroup $\Gamma \le \mathbb{Z}^d$ such that the restriction of $1-\mathbf{1}_A$ to each coset of Γ is $(d-1)$-periodic. Hence the restriction of $\mathbf{1}_A$ to each coset of Γ is $(d-1)$-periodic. Thus, if $u_1,\dots,u_r$ are cosets representatives of Γ in $\mathbb{Z}^d$, setting $A_{u_i} = A\cap(u_i+\Gamma)\subset\mathbb{Z}^d$ yields a decomposition $A = A_{u_1}\uplus\ldots\uplus A_{u_r}$ of A into finitely many $(d-1)$-periodic sets, as required. □

6. From piecewise $(d-1)$-periodicity to d-periodicity

The following lemma extracts an idea that appears within the proof of [GT21a, Theorem 5.4].

Lemma 6.1. *Suppose that $f_1,\dots,f_r,f:\mathbb{Z}^d\to\mathbb{R}$ are bounded functions satisfying $f=\sum_{i=1}^r f_j$. Assume additionally that:*

(1) $\mathrm{stab}(f_i)+\mathrm{stab}(f_j)$ *is a finite index subgroup of $\mathbb{Z}^d$ for all $1\le i<j\le r$.*
(2) $\mathrm{stab}(f)$ *is a finite index subgroup of $\mathbb{Z}^d$.*

Then, for each $1\le j\le r$, the group $\mathrm{stab}(f_j)$ *is of finite index in $\mathbb{Z}^d$.*

Proof. Let $g_1=f_1-f$ and $g_j=f_j$ for $2\le j\le r$. Then $g_1+\ldots+g_r=0$ and $\mathrm{stab}(g_i)+\mathrm{stab}(g_j)$ is a finite index subgroup of $\mathbb{Z}^d$ for all $1\le i<j\le r$. Using the fact that 0 is a polynomial, and applying Lemma 5.1, we get that each g_i is a polynomial with respect to a finite index subgroup of $\mathbb{Z}^d$. But each g_i is bounded. By Lemma 2.2, a polynomial with respect to a finite index subgroup of $\mathbb{Z}^d$ that is bounded must be constant on cosets of this finite index subgroup. This implies that for each $1\le j\le r$ the group $\mathrm{stab}(f_j)$ is of finite index in $\mathbb{Z}^d$. □

Theorem 1.7 is a direct consequence of the above lemma, as shown below.

Proof of Theorem 1.7. Set $f_j=\mathbf{1}_{A_j}$, then $\sum_{j=1}^r f_j = 1$. Let $L_j\le\mathbb{Z}^d$ be the subgroups of rank at least $d-1$ that stabilizes A_j. Note that for every two such subgroups $L_{j_1},L_{j_2}\le\mathbb{Z}^d$, either their intersection has rank $d-1$ or their sum has finite index in $\mathbb{Z}^d$. Assume by contradiction that the intersection of all L_j's is of rank less than $d-1$. By unifying some of the A_j's we can assume without loss of generality that $L_{j_1}+L_{j_2}$ is a finite index subgroup of $\mathbb{Z}^d$ for all $1\le i<j\le r$. In this case, the conditions of Lemma 6.1 hold but the conclusion fails, by the initial assumption. Thus the assumption that $\mathrm{rank}\left(\bigcap_{j=1}^r L_j\right)<d-1$ is false. □

We would also need the following lemma.

Lemma 6.2. *Suppose that $\Sigma\Subset\mathbb{R}$ is a finite set of real numbers, $g_1,\dots,g_r:\mathbb{Z}^d\to\mathbb{R}$ are finitely supported functions and $f:\mathbb{Z}^d\to\Sigma$ is a $(d-1)$-periodic function such that g_j*f is d-periodic for every $1\le j\le r$. Then there exists a d-periodic function $\tilde f:\mathbb{Z}^d\to\Sigma$ such that $g_j*f=g_j*\tilde f$ for every $1\le j\le r$.*

Proof. Consider the space

$$X=\{\tilde x\in\Sigma^{\mathbb{Z}^d} : \forall 1\le j\le r,\ g_j*\tilde x = g_j*f\},$$

and let $\Gamma = \bigcap_{j=1}^r \mathrm{stab}(g_j * f)$. Then X is a Γ-shift of finite type, and by definition $f \in X$ is a $(d-1)$-periodic point in X. Apply Lemma 2.10 to conclude that there exists $\tilde{f} \in X$ that is d-periodic. Any such point $\tilde{f}$ satisfies the conclusion of the lemma. ☐

At this stage, we are prepared to present the proof of Theorem 1.3.

Proof of Theorem 1.3. Suppose that $A \subseteq \mathbb{Z}^d$ is a piecewise $(d-1)$-periodic joint co-tile for $F_1, \dots, F_k \Subset \mathbb{Z}^d$. That is, there exists functions $f_1, \dots, f_r : \mathbb{Z}^d \to \{0,1\}$, each f_j is $(d-1)$-periodic, and $\mathbf{1}_A = \sum_{j=1}^r f_j$. Notice that we may assume that $\mathrm{rank}\left(\bigcap_j \mathrm{stab}(f_j)\right) < d-1$. Indeed, if $\mathrm{rank}\left(\bigcap_j \mathrm{stab}(f_j)\right) \geq d-1$ then $\mathbf{1}_A = \sum_{j=1}^r f_j$ is a $(d-1)$-periodic point in the shift of finite type $\bigcap_{i=1}^k \mathrm{Tile}(F_i;\mathbb{Z}^d)$, and thus by Lemma 2.10 it contains a d-periodic point. Also note that for every two subgroups $L_1, L_2 \leq \mathbb{Z}^d$ having rank at least $d-1$, either their intersection has rank at least $d-1$ or $L_1 + L_2$ has finite index in $\mathbb{Z}^d$. So as before, by possibly summing some of the f_j's we can assume without loss of generality that $\mathrm{stab}(f_l) + \mathrm{stab}(f_j)$ is a finite index subgroup of $\mathbb{Z}^d$ for all $1 \leq l < j \leq r$. Now consider the functions $\mathbf{1}_{F_i} * f_j$. Observe that for every $1 \leq i \leq k$ we have $\sum_{j=1}^r \mathbf{1}_{F_i} * f_j = 1$, and for every $1 \leq j \leq r$ we have $\mathrm{stab}(f_j) \leq \mathrm{stab}(\mathbf{1}_{F_i} * f_j)$. Thus setting $\Lambda_{i,j} := \mathrm{stab}(\mathbf{1}_{F_i} * f_j)$ yields that $\mathrm{rank}\,(\Lambda_{i,j}) \geq d-1$ and $\Lambda_{i,l} + \Lambda_{i,j}$ is a finite index subgroup of $\mathbb{Z}^d$, for every $1 \leq i \leq k$ and $1 \leq l < j \leq r$. Applying Lemma 6.1 for each $1 \leq i \leq k$ separately we see that each $\Lambda_{i,j}$ is a finite index subgroup of $\mathbb{Z}^d$. That is, each one of the functions $\mathbf{1}_{F_i} * f_j$ is d-periodic. For any fixed $1 \leq j \leq r$, applying Lemma 6.2 with $g_i = \mathbf{1}_{F_i}$ and $f = f_j$ and $\Sigma = \{0,1\}$, yields a d-periodic function $\tilde{f}_j : \mathbb{Z}^d \to \Sigma$ that satisfies $\mathbf{1}_{F_i} * \tilde{f}_j = \mathbf{1}_{F_i} * f_j$, for all $1 \leq i \leq k$. In particular, the function $f : \mathbb{Z}^d \to \mathbb{Z}$ defined by $f := \sum_{j=1}^r \tilde{f}_j$ is bounded, d-periodic, and it satisfies

$$\forall 1 \leq i \leq k : \qquad \mathbf{1}_{F_i} * f = \mathbf{1}_{F_i} * \left(\sum_{j=1}^r \tilde{f}_j\right) = \sum_{j=1}^r \mathbf{1}_{F_i} * \tilde{f}_j = \sum_{j=1}^r \mathbf{1}_{F_i} * f_j = 1,$$

Since $\tilde{f} := \sum_{j=1}^r \tilde{f}_j$ is a sum of $\{0,1\}$-valued functions and $1_{F_i} * f = 1$, it follows that f itself is $\{0,1\}$-valued, hence $\tilde{A}$ is an indicator of a set $\tilde{A}$ such that $F_i \oplus \tilde{A} = \mathbb{Z}^d$. Since each $\tilde{f}_j$ is d-periodic, so is $\tilde{A}$. This completes the proof.

☐

7. Constructing independent tiles with the 1-hyperplane repetition property for a periodic co-tile

In this section, we prove Theorem 1.8. We repeatedly rely on the following basic fact.

Lemma 7.1. *Let $L \leq \mathbb{Z}^d$ be a finite index subgroup and let $U_1, \dots, U_r \subset \mathbb{R}^d$ be affine subspaces of dimension strictly smaller than d. Then the set $L \setminus \bigcup_{i=1}^r U_i$ is infinite.*

Proof. For $n \in \mathbb{N}$ let $B_n = \{-n, \dots, n\}^d$. Then there exist $c, c_1, \dots, c_r > 0$ such that $|B_n \cap L| \geq cn^d$ while $|B_n \cap U_i| \leq c_i n^{\dim U_i} \leq c_i n^{d-1}$. In particular, $|B_n \cap (L \setminus \bigcup_{i=1}^r U_i)|$ tends to infinity as n tends to infinity. ☐

Lemma 7.2. *Let $F \Subset \mathbb{Z}^d$, let $A \subseteq \mathbb{Z}^d$ such that $F \oplus A = \mathbb{Z}^d$ and let $L \leq \mathbb{Z}^d$ be a subgroup satisfying $A + L = A$. Then for every function $f : F \to L$ the tile set*

$$F_f := \{v + f(v) \ : \ v \in F\}$$

satisfies $F_f \oplus A = \mathbb{Z}^d$.

Proof. Given a function $f : F \to L$, we show that $F_f \oplus A = \mathbb{Z}^d$. The condition $F \oplus A = \mathbb{Z}^d$ can be rewritten as $\mathbb{Z}^d = \biguplus_{v\in F}(v + A)$. Since $A + L = A$ and $f(v) \in L$ for every $v \in F$, it follows that $f(v) + A = A$. Thus,

$$\mathbb{Z}^d = \biguplus_{v\in F}(v + A) = \biguplus_{v\in F}(v + f(v) + A) = \biguplus_{\tilde{v}\in F_f}(\tilde{v} + A).$$

This proves that $F_f \oplus A = \mathbb{Z}^d$. □

Lemma 7.3. *Suppose we are given* $d, m \in \mathbb{N}$, $(v_1, \ldots, v_m) \in \mathbb{Z}^d$ *and a finite index subgroup* $L \leq \mathbb{Z}^d$. *Given a subset* $J \subseteq \{1, \ldots, m\}$ *and a subspace* $W < \mathbb{R}^d$, *let* $V_W(g, J)$ *denote the subspace of* $\mathbb{R}^d/W$ *obtained by projecting* $\mathrm{span}\{v_j + g(j) : j \in J\}$ *into* $\mathbb{R}^d/W$ *via the map* $v \mapsto v + W$. *Then for every finite collection* $\mathcal{W}$ *of proper subspaces of* $\mathbb{R}^d$ *there exists a function* $g : \{1, \ldots, m\} \to L$ *so that for every* $J \subseteq \{1, \ldots, m\}$ *and every* $W \in \mathcal{W}$ *we have*

$$\dim (V_W(g, J)) = \min\{d - \dim(W), |J|\}.$$

Proof. We prove the claim by induction on m. For $m = 1$, we only need to choose $g(1) \in L$ such that $v_1 + g(1) \notin W$ for any $W \in \mathcal{W}$. This is possible by Lemma 7.1. Assume by induction that $g(1), \ldots, g(m) \in L$ have been defined so that the conclusion holds for every $J \subseteq \{1, \ldots, m\}$ and every $W \in \mathcal{W}$. Using Lemma 7.1 we can choose $g(m+1) \in L$ that is not contained in any affine hyperplane of the form $U := -v_{m+1} + \mathrm{span}\{v_j + g(j) : j \in J\} + W$, where $W \in \mathcal{W}$ and J ranges over subsets of $\{1, \ldots, m\}$ of size at most $d - \dim(W) - 1$. We need to show that for any $J \subseteq \{1, \ldots, m + 1\}$ and $W \in \mathcal{W}$ we have $\dim (V_W(g, J)) = \min\{d - \dim(W), |J|\}$. Fix some $J \subseteq \{1, \ldots, m + 1\}$ and $W \in \mathcal{W}$. The assertion follows from the induction hypothesis in case $(m + 1) \notin J$, so suppose $(m + 1) \in J$. By the induction hypothesis, $\dim (V_W(g, J \setminus \{m + 1\})) = \min \{d - \dim(W), |J \setminus \{m + 1\}|\}$. If $|J\setminus\{m+1\}| \geq d-\dim(W)$, then $\dim_W(V(g, J)) = d - \dim(W)$, as required. Otherwise, we have that

$$\dim(V_W(g, J \setminus \{m + 1\})) = |J \setminus \{m + 1\}| = |J| - 1.$$

By our choice of $g(m+1)$, we have that $v_{m+1}+g(m+1) \notin \mathrm{span}\,\{v_j + g(v_j) \;:\; j \in J \setminus \{m + 1\}\}$, so

$$\dim_W(V(g, J)) = \dim(V_W(g, J \setminus \{m + 1\})) + 1 = |J|.$$

This completes the induction step, hence the proof. □

Proof of Theorem 1.8. Suppose $F \oplus A = \mathbb{Z}^d$ where $L \in \mathbb{Z}^d$ is a finite index subgroup satisfying $A + L = A$. Write $F^* = \{w_1, \ldots, w_k\}$. We apply Lemma 7.3 with $m = (d - 1)k$ and $(v_1, \ldots, v_m)$, where $v_{kj+i} = w_i$ for $0 \leq j \leq d-2$, and $1 \leq i \leq k$, and $\mathcal{W} = \{\mathrm{span}\{v\} \;:\; v \in F\}$ to obtain a function $g : \{1, \ldots, m\} \to L$ as in the statement of Lemma 7.3. For $0 \leq j \leq d-2$ we set

$$F_{j+1} = \{0\} \cup \{v_{kj+i} + g(kj + i) : 1 \leq i \leq k\} = \{0\} \cup \{w_i + g(kj + i) : 1 \leq i \leq k\}.$$

By Lemma 7.2 we indeed have $F_j \oplus A = \mathbb{Z}^d$ for every $1 \leq j \leq d - 1$. To see that $(F_1, \ldots, F_{d-1}, F)$ is a d-tuple of independent tiles, note that for any choice of $(u_1, \ldots, u_{d-1}, v) \in F_1^* \times \ldots \times F_{d-1}^* \times F$ there exists $i_1, \ldots, i_{d-1} \in \{1, \ldots, k\}$ so that

$$u_j = v_{k(j-1)+i_j} + g(k(j - 1) + i_j).$$

Hence, there exists a set $J \subset \{1, \ldots, k(d - 1)\}$ so that

$$\mathrm{span}\{u_1 + W, \ldots, u_{d-1} + W\} = V_W(g, J),$$

where $W = \text{span}\{v\}$. By the property of g, it follows that $\dim(\text{span}\{u_1, \dots, u_{d-1}, v\}) = d$.

Let us check that $\{F_1, \dots, F_{d-2}, F\}$ has the property $(\star)$. Choose two distinct $(d-2)$-tuples

$$(u_1, \dots, u_{d-2}), (\tilde{u}_1, \dots, \tilde{u}_{d-2}) \in F_1^* \times \dots \times F_{d-2}^*,$$

and $v, \tilde{v} \in F$. As before, it follows that there exists subsets $J, \tilde{J} \subset \{1, \dots, m\}$ with $J \neq \tilde{J}$ and $|J| = |\tilde{J}| = d - 2$ so that

$$\{u_1, \dots, u_{d-2}\} = \{v_j : \ j \in J\} \text{ and } \{\tilde{u}_1, \dots, \tilde{u}_{d-2}\} = \{v_j \ : \ j \in J'\}.$$

Since $J \neq \tilde{J}$ and $|J| = |\tilde{J}| = d - 2$, there exists $\ell \in \tilde{J} \setminus J$. It follows from the property of the function g that for any $v \in F^*$

$$\dim(\text{span}(\{v_j \ : \ j \in J\} \cup \{v\}) = d - 1$$

and

$$\dim(\text{span}(\{v_j \ : \ j \in J\} \cup \{v\} \cup \{v_\ell\}) = d.$$

This shows that $v_\ell \notin \text{span}\left(\{v_j \ : \ j \in J\} \cup \{v\}\right)$. In particular, there does not exist $v \in F^*$ such that

$$\text{span}\left(\{\tilde{u}_1, \dots, \tilde{u}_{d-2}\}\right) \subseteq \text{span}\left(\{u_1, \dots, u_{d-2}, v\}\right).$$

This shows that there does not exist $v, \tilde{v} \in F^*$ so that

$$\text{span}\left(\{\tilde{u}_1, \dots, \tilde{u}_{d-2}, \tilde{v}\}\right) = \text{span}\left(\{u_1, \dots, u_{d-2}, v\}\right),$$

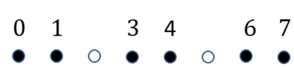
which proves that $(F_1, \dots, F_{d-2}, F)$ has property $(\star)$. □

8. Further comments and questions

8.1. **Integer-valued co-tiles.** Given $F \Subset \Gamma$, we say that a bounded function $f : \Gamma \to \mathbb{Z}$ is an *integer-valued co-tile* for F if $\mathbf{1}_F * f = 1$. Observe that our proof of Theorem 1.3 holds for integer-valued co-tile as well, thus we have:

Proposition 8.1. *Let k and d be positive integers and let $F_1, \dots, F_k \Subset \mathbb{Z}^d$. Suppose that $F_1, \dots, F_k$ admit an integer-valued joint co-tile f and that $f = \sum_{i=1}^r f_r$, where each $f_i : \mathbb{Z}^d \to \mathbb{Z}$ is bounded and $(d-1)$-periodic. Then $F_1, \dots, F_k$ admit a d-period integer-valued joint co-tile.*

It is natural to ask whether the existence of an integer-valued co-tile for $F \Subset \Gamma$ implies the existence of a set $A \subseteq \Gamma$ for which $\mathbf{1}_F * \mathbf{1}_A = 1$? The simple example below shows that this is not true even for $\Gamma = \mathbb{Z}$ (or for Γ a finite cyclic group, here $\mathbb{Z}/18\mathbb{Z}$). Let $F_1 = \{0, 1\}$, $F_2 = \{0, 3, 6\}$ and $F = F_1 \oplus F_2 = \{0, 1, 3, 4, 6, 7\}$.



We claim that F does not tile $\mathbb{Z}$, but it does admit an integer-valued co-tile. Note that for $A_1 = 2\mathbb{Z}$ and $A_2 = \{0, 1, 2\} \oplus 9\mathbb{Z}$ we have

$$F_1 \oplus A_1 = F_2 \oplus A_2 = \mathbb{Z}.$$

Furthermore, if $\tilde{A}_1$ is a co-tile for F_1 then $\tilde{A}_1$ must be a translate of A_1. To see that F does not tile $\mathbb{Z}$, suppose by contradiction that $F \oplus A = \mathbb{Z}$ then $F_1 \oplus (F_2 \oplus A) = \mathbb{Z}$, so we must have that $F_2 \oplus A$ is a coset of $2\mathbb{Z}$, but this is clearly impossible since F_2 is not contained in a coset of $2\mathbb{Z}$. Now take

$$f = \mathbf{1}_{A_1} - \mathbf{1}_{A_2}.$$

Then using $\mathbf{1}_F = \mathbf{1}_{F_1} * \mathbf{1}_{F_2}$ and $\mathbf{1}_{F_i} * 1 = |F_i|$ we get:

$$\mathbf{1}_F * f = \mathbf{1}_{F_2} * (\mathbf{1}_{F_1} * \mathbf{1}_{A_1}) - \mathbf{1}_{F_1} * (\mathbf{1}_{F_2} * \mathbf{1}_{A_2}) = |F_2| - |F_1| = 1.$$

8.2. **Conditions for joint tilings for d independent tiles in $\mathbb{Z}^d$.** In view of Theorem 1.2, the classical Wang argument (see [Ber66], [Rob71]) implies that it is algorithmically decidable whether a set of d independent tiles in $\mathbb{Z}^d$ admit a joint co-tile: Indeed, any such tiling must be periodic so we can exhaust the possible periodic co-tiles. As in [GT21a], from an upper bound for the period of a co-tile one can directly deduce an upper bound for the computational complexity of the tiling problem. It is of interest to find explicit necessary and sufficient conditions for a d-tuple of independent subsets of $\mathbb{Z}^d$ to admit a joint co-tile. In view of Theorem 1.8, the previous problem is closely related to the more basic question of finding explicit necessary and sufficient conditions for a finite set of $\mathbb{Z}^d$ to tile periodically.

Conversely, one can ask about necessary and sufficient conditions for an infinite subset of $\mathbb{Z}^d$ to be a joint co-tile for d-independent tiles. In view of Theorem 1.2 and Theorem 1.8 this is equivalent to the question of finding necessary and sufficient conditions for a periodic subset of $\mathbb{Z}^d$ to be a co-tile for a finite tile.

A complete solution to the above questions involves the factorization of finite abelian groups, namely understanding solutions for $A \oplus B = G$, where G is a finite abelian group. This is a difficult problem even in the cyclic case $G = \mathbb{Z}/M\mathbb{Z}$, which comes up in tilings of $\mathbb{Z}$.

Coven and Meyerowitz [CM99] found explicit and efficiently verifiable sufficient conditions for tiling the integers by a finite set. It has been conjectured that these conditions are also necessary. This conjecture has been verified in some specific cases recently [ŁL22a, ŁL22b]. The necessity of the Coven-Meyerowitz conditions would imply an efficient algorithm for determining if a given finite subset $F \Subset \mathbb{Z}$ can tile $\mathbb{Z}$, see [KM09].

8.3. **Higher level tilings.** A level ℓ co-tile of $\mathbb{Z}^d$ by a finite set set $F \Subset \mathbb{Z}^d$ is a set $A \subseteq \mathbb{Z}^d$ such that $\mathbf{1}_F * \mathbf{1}_A = \ell$. Both Theorem 1.1 and Theorem 1.2 generalize to level ℓ tilings. A suitable modification of Proposition 2.5 implies that if $\mathbf{1}_F * f = \ell$ then f has mean $\frac{\ell}{|F|}$. A proof can be obtained via a relatively routine modification of Theorem 3.1 as follows:

Theorem 8.2. *Let $\ell_1, \dots, \ell_k \in \mathbb{N}$, $F_1, \dots, F_k \Subset \mathbb{Z}^d$, with $0 \in F_i$ for all $1 \le i \le k$, and let $f : \mathbb{Z}^d \to \mathbb{Z}$ be a bounded function that satisfies $\mathbf{1}_{F_i} * f = \ell_i$ for all $1 \le i \le k$. Then for every $1 \le i \le k$ and every $(v_1, \dots, v_i) \in F_1^* \times \ldots \times F_i^*$ there exists a function $\phi_{v_1,\dots,v_i} : \mathbb{Z}^d \to [\min f, \max f]$ with the following properties:*

(a) For $i < k$ we have

$$\phi_{v_1,\dots,v_i} = \ell_{i+1} - \sum_{v_{i+1} \in F_{i+1}} \phi_{v_1,\dots,v_i,v_{i+1}}.$$

(b)

$$f = (-1)^i \sum_{(v_1,\dots,v_i) \in F_1^* \times \ldots \times F_i^*} \phi_{v_1,\dots,v_i} + \sum_{j=1}^{i} (-1)^{j-1} \prod_{t=1}^{j} \ell_t \prod_{s=1}^{j-1} |F_s|.$$

(c) Let q denote the product of all primes less than or equal to $\max_{1 \le i \le k} \ell_k (\max f - \min f) \max_{1 \le i \le k} |F_i|$, then

$$(\mathbb{Z}qv_1 + \ldots + \mathbb{Z}qv_i) \le \operatorname{stab}(\phi_{v_1,\dots,v_i}),$$

(d) $\mathbf{1}_{F_j} * \phi_{v_1,\dots,v_i} = \ell_i$ for every $1 \le j \le k$. In particular, it has mean $\ell_i/|F_i|$.

8.4. **Piecewise 1-periodicity of co-tiles in $\mathbb{Z}^2 \times (\mathbb{Z}/p\mathbb{Z})$.** By applying the arguments of Section 4, the methods of [GT21a] directly give:

Theorem 8.3. *Let p be a prime number, $\Gamma = \mathbb{Z}^2 \times (\mathbb{Z}/p\mathbb{Z})$ and $F \Subset \Gamma$ be a finite set. Then one of the following holds:*

(1) Any $A \subset \Gamma$ satisfying $F \oplus A = \Gamma$ is piecewise 1-periodic.
(2) There exist a finite set $\tilde{F} \subset \mathbb{Z}^2$ such that $F = \tilde{F} \times (\mathbb{Z}/p\mathbb{Z})$.

In fact, using Theorem 4.4 and the results of Section 5, we can deduce the following: For any rank 2 abelian group Γ and any $F \Subset \Gamma$, if $F \oplus A = \Gamma$ then the set $F^{\mathrm{Tor}} \oplus A$ is piecewise 1-periodic, whereas in Section 4, F^{Tor} is the intersection of F with the torsion subgroup of Γ. Then in the case $\Gamma = \mathbb{Z}^2 \times \mathbb{Z}/p\mathbb{Z}$ with p prime, Lemma 4.5 implies Theorem 8.3.

Corollary 8.4. *Let p be a prime number, $\Gamma = \mathbb{Z}^2 \times (\mathbb{Z}/p\mathbb{Z})$ and $F \Subset \Gamma$ be a finite set. If F tiles Γ, then F tiles Γ periodically.*

Rachel Greenfeld and Terence Tao have informed us in private communication that they also obtained Corollary 8.4.

8.5. **A Fourier-analytic and algebraic-geometric approach.** Fourier analytic methods are a natural approach to translational tiling problems, see [GT21a, Remark 1.8]. Let $g_1, \dots, g_d : \mathbb{Z}^d \to \mathbb{C}$ be finitely supported functions, by which we mean that $g_i(v) = 0$ for all but finitely many $v \in \mathbb{Z}^d$. Suppose $f : \mathbb{Z}^d \to \mathbb{C}$ is a bounded function that satisfies $g_i * f = 1$ for all $1 \le i \le d$. Taking distributional Fourier transform on both sides yields

$$\hat{g}_i \cdot \hat{f} = \delta_0.$$

Thus, the distributional Fourier transform of f is supported on 0 and the intersection of the zeros of $\hat{g}_i$. In particular, if $\hat{g}_1, \dots, \hat{g}_d$ have finitely many common zeros, and f must be the Fourier transform of a multivariate trigonometric polynomial, hence periodic.

The set of common zeros for d polynomials in d variables is "generically" a finite set. Given $v = (n_1, \dots, n_d) \in \mathbb{Z}_+^d$ let $X^v := x_1^{n_1} \cdot \dots \cdot x_d^{n_d}$ denote the corresponding monomial in d variables $x_1, \dots, x_d$. Given a finite set $F \Subset \mathbb{Z}_+^d$, let $P_F := \sum_{v \in F} X^v$ denote the corresponding multivariate polynomial. We conclude that whenever $F_1, \dots, F_d \Subset \mathbb{Z}_+^d$ are subsets such that the algebraic variety

$$V(P_{F_1}, \dots, P_{F_d}) := \bigcap_{i=1}^{d} \left\{ (x_1, \dots, x_d) \in \mathbb{C}^d \ : \ P_{F_i}(x_1, \dots, x_d) = 0 \right\}$$

has a finite intersection with the d-sphere, then any joint co-tile for $F_1, \dots, F_d$ is periodic.

This raises the question: Is it true that for an independent d-tuple $(F_1, \dots, F_d)$ in $\mathbb{Z}^d$ the algebraic variety $V(P_{F_1}, \dots, P_{F_d})$ is finite?

We note that it can be shown that $V(P_{F_1}, \dots, P_{F_d})$ is finite if we impose the somewhat stronger condition that $(F_1 - F_1, \dots, F_d - F_d)$ is an independent d-tuple in $\mathbb{Z}^d$. This follows from the equality of the tropical variety with the Bieri-Groves set of the variety (see Theorem 2.2.5 and Corollary 2.2.6 in [EKL06]), combined with [EKL06, Theorem 2.2.3] and an explicit direct computation. This connection was kindly explained to us by Ilya Tyomkin. This argument gives an alternative derivation of the conclusion of Theorem 1.2, under the slightly stronger assumption that $(F_1 - F_1, \dots, F_d - F_d)$ is an independent tuple of tiles on $\mathbb{Z}^d$.

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