Discrete Subsets and Tilings of Euclidean Spaces

Thesis submitted in partial fulfillment
of the requirements for the degree of
"DOCTOR OF PHILOSOPHY"

by

Yaar Solomon

Submitted to the Senate of Ben-Gurion University
of the Negev

May 31st, 2013.

Beer-Sheva

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Approved by the advisor
Approved by the Dean of the Kreitman School of Advanced Graduate Studies

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This work was carried out under the supervision of

Professor Barak Weiss.

In the Department of Mathematics.
Faculty of Science.

Research-Student's Affidavit when Submitting the Doctoral Thesis for Judgment

I, Yaar Solomon, whose signature appears below, hereby declare that:

_X_ I have written this Thesis by myself, except for the help and guidance offered by my Thesis Advisors.

_X_ The scientific materials included in this Thesis are products of my own research, culled from the period during which I was a research student.

__ This Thesis incorporates research materials produced in cooperation with others, excluding the technical help commonly received during experimental work. Therefore, I am attaching another affidavit stating the contributions made by myself and the other participants in this research, which has been approved by them and submitted with their approval.

Date:____________

Student's name:__Yaar Solomon__

Signature:____________

To Naama, my wife.

List of Symbols

Abbreviation	Description Definition
$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	Natural numbers, integers, rational, real, and complex numbers.
$d(\cdot, \cdot)$	The Euclidean metric.
$d_1(\cdot, \cdot), d_\infty(\cdot, \cdot)$	The ℓ_1, and ℓ_∞ metrics.
$\|\cdot\|$	The Euclidean norm.
$\|\cdot\|_1, \|\cdot\|_\infty$	The ℓ_1, and ℓ_∞ norms.
$B(x, r)$	The Euclidean ball of radius r around x.
$B_1(x, r), B_\infty(x, r)$	The balls w.r.t. the ℓ_1, and ℓ_∞ norms.
$diam(A)$	The diameter of a set A, w.r.t. $d(\cdot, \cdot)$.
$int(A), \partial A, \overline{A}$	The interior, boundary, and closure of a set A.
$\lambda(A)$ or $Vol(A)$	The Lebesgue measure of a set A.
$\lambda_s(A)$	The s-dimensional Lebesgue measure of a set $A \subseteq \mathbb{R}^d$
a.e.	almost everywhere w.r.t. Lebesgue measure
$Jac(\varphi)(x)$	The Jacobian determinant of φ at x
$biLip(\varphi)$	The smallest (inf) biLipschitz constant of φ

Contents

Abstract

We study properties of infinite discrete subsets of Euclidean spaces. The basic objects that we study are called separated nets, which are uniformly discrete and relatively dense subsets of $\mathbb{R}^d$. There is a correspondence between separated nets and tilings of $\mathbb{R}^d$ by finitely many translation equivalent tiles. This correspondence matches between tilings and separated nets, up to a bounded displacement (BD) of the net, namely moving each point a bounded distance. A significant part of our research deals with three equivalence relations on the set of separated nets: BD equivalence, BD equivalence after dilation, and biLipschitz equivalence, where BD equivalence is the most delicate of these three. Considering the above correspondence between tilings and separated nets, which is up to BD equivalence, it suffices to consider separated nets that come from tilings when studying those three equivalence relations.

Most of the questions that we answer in this dissertation are trivial for periodic tilings. We focus on a specific family of tilings that are called substitution tilings. These are tilings with finitely many translation equivalent tiles, that possess strong self-similarity properties, and many of their qualities can be derived out of them. These tilings are generated by a process of substituting each tile by some tessellation, using the same type of tiles, but with a smaller scale. Repeating those substitutions over and over again produces tilings of larger and larger regions, which as limits give rise to tilings of the whole space. The limiting objects are called substitution tilings. One of the reasons to study these tilings is that they are usually non-periodic, and in fact this is one of a very few procedures that we have to produce non-periodic tilings.

In §2 we answer a question that we were asked privately by McMullen, which was motivated by the results of the author MSc thesis, on biLipschitz equivalence of separated nets that arise from substitutions. The results of this chapter are in [S11b]. As it was shown by McMullen in [McM98], and by Burago and Kleiner in [BK98], there is a tight connection between the following two questions:

(1) Given a separated net $Y \subseteq \mathbb{R}^d$, is it biLipschitz to $\mathbb{Z}^d$?

(2) Given a positive function $f \in L^\infty(\mathbb{R}^d)$, is it the Jacobian of a biLipschitz homeomorphism of $\mathbb{R}^d$ a.e.?

Given a tiling τ of $\mathbb{R}^d$ consider the function $f_\tau : \mathbb{R}^d \to \mathbb{R}$, which is constant $1/Vol(T)$ on the interior of every tile T. It is not difficult to show that wherever f_τ is a Jacobian as in (2), any separated net that corresponds to τ is biLipschitz to $\mathbb{Z}^d$. We show that for substitution tilings τ, f_τ is a Jacobian wherever a local condition is satisfied for each of the basic tiles. We also show that our condition is satisfied for tilings by star-shaped domains in the plane, and therefore the theorem holds for such tilings, and in particular f_τ is a Jacobian for every Penrose Tiling τ.

Chapter 3 deals with the BD equivalence relation on separated nets that arise from substitution tilings of $\mathbb{R}^d$, and describes the results of [S12]. BD equivalence of separated nets implies biLipschitz equivalence, which is an equivalence relation that was vastly studied, for instance in [BK98], [BK02], [McM98], [Gr93]. The study of the BD equivalence relation began at the author's MSc thesis, see [S11a], and attracted other authors to consider it, see [ACG11], [HKW12], and [S12]. We study separated nets that corresponds to substitution tilings, and ask when such a net is BD to a lattice? This question was considered before in [S11a] and [ACG11], and our results here improve these previous results and answer the question for almost all possible cases. Moreover, we show that the answer of whether the net is BD to a lattice or not does not depend on the tiling itself, but only on the properties of the dissection rule of the basic tiles, or more specifically the eigenvalues and eigenspaces of the matrix that it defines. The main step that allows the improvement in our proof here is Proposition 3.3.5. By the nature of the definition of substitution tilings, if τ is such a tiling and H is the substitution rule on the tiles, then for every $m \in \mathbb{N}$ there is a tiling τ_m with $H^m(\tau_m) = \tau$. In Proposition 3.3.5 we prove that any finite patch P in τ can be presented as unions and proper differences of tiles from different generations τ_m, with good estimates on the number of tiles taken from every generation. Using these estimates we can apply a theorem of Laczkovich from [L92] and deduce the conclusion on the BD equivalence.

In Chapter 4 we work on the Danzer problem, which is an open problem for nearly sixty years already. There are several different phrasings for this problem, and the following is the original one, which is due to Danzer: Is there a set $D \subseteq \mathbb{R}^d$, with growth rate $O(T^d)$, that intersects every convex set of volume 1? Despite the fact that this question can be presented in one line, and no prior knowledge in required to approach it, the only previous result that we know of is in [BW71]. In that paper Bambah and Woods have one negative and one positive results regarding Danzer sets, namely sets that intersect all convex sets of volume 1. They first show that any finite union of grids cannot be a Danzer set, and then they construct a Danzer set in $\mathbb{R}^d$ with growth rate $O(T^d(\log T)^{d-1})$. We also have here one negative and one positive result. We show that discrete sets that arise from substitution tilings of $\mathbb{R}^d$ by choosing one point in each basic tile are not Danzer sets. Then we show that the Danzer problem is actually equivalent to a well-known, difficult, combinatorial problem, and use partial result of that problem to deduce a construction of a Danzer set in $\mathbb{R}^d$ of growth rate $O(T^d \log T)$.

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Chapter 1

Introduction

This dissertation deals with different properties of tilings and infinite discrete subsets of Euclidean spaces. We usually focus on non-periodic discrete sets, and try to understand properties which are usually obvious for the periodic case. There is a simple duality between tilings and discrete subsets of Euclidean spaces, and it turns out to be efficient to use properties and classifications of one for studying the other. One of our main directions is to study discrete subsets that correspond to families of non-periodic tilings. That way we get the non-periodicity on one hand, and have additional interesting structure on the other.

1.1 History and Motivation

One of the motivations for studying these objects is the work of Shechtman et al [SBGC84] on quasicrystal with no translational symmetry. The patterns that are obtained in an electron diffraction of these materials are nice examples for the discrete point sets that we are looking at, and this work shed more light on these materials from the mathematical point of view. Another reason for this research is to understand better the well known Penrose Tiling, see for instance [DeB81], [Gar77], [GS87], [P74], [P79]. This was the one of first examples of a non-periodic tiling of the plane, and it uses only two non-isometric tiles. Our results of the following three chapters also hold specifically for the Penrose Tiling. One other motivation comes from Geometric Group Theory, where metric spaces are sometimes considered up to the equivalence relation of quasi-isometry. Our research focuses on separated nets, sets of points which are both uniformly discrete and relatively dense, and it can be shown that two metric spaces are quasi-isometric if and only if they contain biLipschitz equivalent separated nets. This observation gives rise to the following question by Gromov, see [Gr93], p. 23: Is every separated net in $\mathbb{R}^d$ biLipschitz equivalent to $\mathbb{Z}^d$ $(d > 1)$? The themes that are discussed in Chapters 2 and 3 are closely related to this question. One last motivation to study infinite discrete sets is the connections to questions in combinatorics. As we show in Chapter 4, the question that we deal with there is equivalent to a difficult question in combinatorics and computational geometry, and might also have applications in machine learning.

1.2 Separated Nets and BiLipschitz Equivalence

We denote by $\mathbb{R}^d$ the d-dimensional Euclidean space, and by $d(\cdot,\cdot)$ the Euclidean metric on it.

Definition 1.2.1. A set $Y \subseteq \mathbb{R}^d$ is *uniformly discrete* if there exists $r > 0$ such that for any $y_1, y_2 \in Y$ we have $d(y_1, y_2) \geq r$. It is *relatively dense* if there exists $R > 0$ such that for any $x \in \mathbb{R}^d$ there is some $y \in Y$ with $d(x, y) \leq R$. Y is called a *separated net*, or a *Delone set*, if it is both uniformly discrete and relatively dense.

The first example of a separated net in $\mathbb{R}^d$ is $\mathbb{Z}^d$, or a bit more generally, lattices. As mentioned above, our work is motivated by Gromov's question: Is every separated net in $\mathbb{R}^d$ biLipschitz equivalent to $\mathbb{Z}^d$? Where two separated nets Y_1 and Y_2 are called *biLipschitz equivalent* if there exists a constant $K \geq 1$ and a bijection $\varphi : Y_1 \to Y_2$ that satisfies

$$\frac{1}{K} \leq \frac{d(\varphi(y), \varphi(y'))}{d(y, y')} \leq K,$$

for every two distinct $y, y' \in Y_1$. This question was settled in 1998 by Burago and Kleiner [BK98], and independently by McMullen [McM98], who showed that the answer is negative. In their proof they relate the above question of Gromov to another question with a more analytic nature: Is every positive, L^∞ function $f : \mathbb{R}^d \to \mathbb{R}$ can be realized as a Jacobian of a biLipschitz homeomorphism of $\mathbb{R}^d$ a.e.? They show that those two questions are equivalent, and then explicitly construct a function which is not the Jacobian of any biLipschitz homeomorphism of $\mathbb{R}^d$. This already implies the existence of a separated net that is not biLipschitz to $\mathbb{Z}^d$. Nevertheless, in [BK98] there is an additional part showing how to explicitly construct such nets, given a function which cannot be realize as such a Jacobian. We return to deal with the Jacobian question with more details in Chapter 2, focusing on a more specific type of functions.

Given the fact that not every separated net is biLipschitz equivalent to $\mathbb{Z}^d$, people begin to consider certain constructions of separated nets, and ask whether their special nets always biLipschitz to $\mathbb{Z}^d$. Alternatively, is there a separated net which is not biLipschitz to $\mathbb{Z}^d$ that comes from some nice general construction?

1.3 Tilings and Separated Nets

Lattices in $\mathbb{R}^d$, namely discrete co-compact subgroups, and their translations, are the most basic example of separated nets. It is easy to see that if $\Gamma \subseteq \mathbb{R}^d$ is a lattice, then $\Gamma = A \cdot \mathbb{Z}^d$ for some invertible matrix A. Since invertible linear maps are biLipschitz homeomorphisms, every lattice is biLipschitz equivalent to $\mathbb{Z}^d$. To talk about other examples of separated nets, we present now the connection between these objects and tilings of Euclidean spaces.

A *tile* $T \subseteq \mathbb{R}^d$ is just a subset of $\mathbb{R}^d$, that usually carry a few other regularity properties, that depend on the context. A *tiling*, or a *tessellation*, τ of a subset $S \subseteq \mathbb{R}^d$ is a collection of tiles that covers S, where different tiles intersect only at their boundaries. A tiling P of a bounded set $S \subset \mathbb{R}^d$ is called a *patch*. We call the set S the *support of* P and we denote it by *supp*(P). Given a collection of tiles $\mathcal{F}$, we denote by $\mathcal{F}^*$ the set of all patches by the elements of $\mathcal{F}$. People often consider tiles which are homeomorphic to closed balls, or even polytopes, but tilings by less regular shapes, like disconnected sets, non-simply connected sets, and fractals also appears in many different areas of mathematics. For further reading on tiling see for instance [GS87] and [Ra99].

Given a separated net $Y \subseteq \mathbb{R}^d$, one gets a tiling ν_Y of $\mathbb{R}^d$ by convex polytopes by taking the Voronoi cells

$$\forall y \in Y : \quad C_y = \{x \in \mathbb{R}^d : \forall y' \in Y \setminus \{y\}, d(x,y) \leq d(x,y')\}.$$

Since $H_{y,y'} = \{x \in \mathbb{R}^d : d(x,y) \leq d(x,y')\}$ is a half-space, each C_y is an intersection of half-spaces and therefore convex. It follows from the discreteness of Y that each C_y is defined by finitely many $H_{y,y'}$, and hence a convex polytope.

On the other hand, if τ is a tiling of $\mathbb{R}^d$, picking one point in each tile may produce a separated net Y_τ. Assuming only that the tiles of τ has bounded diameters, and that the inradii are bounded from below, implies that Y_τ is indeed a separated net, as long as we do not pick the points too close to the boundaries. This last condition can be ensured if we allow moving the points inside the tiles. So up to this bounded displacement of the points we obtain a separated net Y_τ. This brings us to the following definitions:

Definition 1.3.1. Let $Y_1, Y_2 \subseteq \mathbb{R}^d$ be two separated nets. We say that Y_1 *is a bounded displacement (BD) of* Y_2 if there exists a bijection $\varphi : Y_1 \to Y_2$ with

$$\sup_{y \in Y_1} \{d(y, \varphi(y))\} < \infty.$$

Y_1 *is a bounded displacement after dilation (BDD) of* Y_2 if there is an $\alpha > 0$ such that Y_1 is BD to $\alpha \cdot Y_2$.

Observe that definition 1.3.1 gives rise to two more equivalence relations on the set of separated nets in $\mathbb{R}^d$, and that

$$\text{BD equivalence} \Longrightarrow \text{BDD equivalence} \Longrightarrow \text{biLipschitz equivalence.}$$

In view of the above, Y is BD to Y_{ν_Y} for any separated net Y, and thus we have a correspondence between tilings and separated nets, up to BD of the nets. Notice that for an arbitrary separated net Y, the tiling ν_Y consists of infinitely many non-isometric tiles. This point can be easily fixed, as the following Proposition shows.

Claim 1.3.2. *For every separated net $Y \subseteq \mathbb{R}^d$ there exists a tiling τ_Y of $\mathbb{R}^d$ by finitely many translation equivalent tiles such that every tile intersect Y in a single point, and Y_{τ_Y} is BD to Y.*

Proof. Let $r > 0$ be such that $d(y_1, y_2) > r$ for every $y_1, y_2 \in Y$. Divide $\mathbb{R}^d$ to dyadic cube Q with edge length less than $r/\sqrt{d}$. Then each cube contains at most one point of Y. For every $y \in Y$ define

$$T_y = \bigcup \left\{ Q : \forall y' \in Y \smallsetminus \{y\}, d(Q, y) < d(Q, y') \right\}.$$

Cubes with equal minimal distance to several points y can be added arbitrarily to one of the corresponding T_y, or alternatively, we may assume that it never happen by replacing Y with a small translation of it. Call the resulting tiling τ_Y. Then clearly $Y \cap T_y = \{y\}$, for all $y \in Y$. Since Y is relative dense, there is an $R > 0$ such that for every y we have $T_y \subseteq B(y, R)$. Each T_y is a union of cubes Q, and there are only finitely many such configuration in a ball of radius R. □

We deduce that when studying either of the *BD*, *BDD*, or *biLipschitz* equivalence classes of separated nets it suffices to look at nets that correspond to tilings with finitely many tiles, up to translations. In particular, the results in [BK98] and [McM98] imply that there are tilings of $\mathbb{R}^d$, with finitely many tiles, that give rise to separated nets which are not even biLipschitz to $\mathbb{Z}^d$. So from now on, unless indicated otherwise, a tiling would mean a tiling by finitely many translation equivalent tiles, which will be denoted by $T_1, \ldots, T_n$. These $T_1, \ldots, T_n$ are called *prototiles*, and these are representatives of the equivalence classes on the tiles of τ, where two tiles are equivalent if they differ by a translation.

The following claims give a better intuition, and explain some basic facts, regarding the BD equivalence relation and tilings. One of the main tools that are used to prove BD equivalence is the following well known theorem, see [H48], [Ra49].

Theorem 1.3.3 (Hall's Marriage Theorem)**.** *Let $G = (A \cup B, E)$ be a locally finite, bipartite graph, and assume that for every finite set $X \subseteq A$ we have*

$$N(X) := \#\{b \in B : \exists x \in X, \{x, b\} \in E\} \geq \#X. \tag{1.1}$$

Then there is an injection $f : A \to B$ such that $\{a, f(a)\} \in E$ for every $a \in A$.

Remark 1.3.4.

- *The condition $N(X) \geq \#X$ for every $X \subseteq A$ is called the* Hall's condition*. This is in fact an 'if and only if' condition for such an injection $f: A \to B$, but the 'only if' part is trivial.*
- *If the Hall's condition holds for every $X \subseteq A$ and every $X \subseteq B$ then there is a bijection $f: A \to B$ with $\{a, f(a)\} \in E$ for every $a \in A$. This can be obtained simply by applying the theorem twice, and repeating the proof of the Cantor-Bernstein Theorem.*
- *The infinite version follows from the finite one. One way to prove it is by using the compactness theorem in first order logic.*

Claim 1.3.5. *Let τ be a tiling of $\mathbb{R}^d$ with either finite or infinite non-equivalent tiles, of bounded diameter. Assume that $Vol(T) = \alpha$ for every tile T, then every Y_τ is BD to $\alpha^{1/d}\mathbb{Z}^d$.*

Proof. We may assume that $\alpha = 1$. Given Y_τ, every point $y \in Y_\tau$ corresponds to a unique tile $T_y \in \tau$. For points $n \in \mathbb{Z}^d$, denote by Q_n the unit cube centered at n. Consider the bipartite graph $G = (Y_\tau \cup \mathbb{Z}^d, E)$, where $(y, n) \in E \overset{\text{iff}}{\Longleftrightarrow} T_y \cap Q_n \neq \varnothing$. We claim that the Hall's condition is satisfied. Take $X = \{y_1, \ldots, y_k\} \in Y_\tau$, and assume that $N(X) < k$. Then $\bigcup_i T_{y_i}$ is covered by less than k cubes Q_n, contradicting the fact that $Vol(T_y) = Vol(Q_n) = 1$ for every y and n. Similarly we get it for every $X \subseteq \mathbb{Z}^d$, and hence by the Hall's Marriage Theorem we get a bijection $f: Y_\tau \to \mathbb{Z}^d$ with $(y, f(y)) \in E$ for every $y \in Y_\tau$. By the definition of E, for every $y \in Y_\tau$ we have $d(y, f(y)) \leq \sup_{y \in Y_\tau}\{diam(T_y)\} + \sqrt{d} < \infty$. □

Corollary 1.3.6. *Every two lattices $\Gamma_1, \Gamma_2 \subseteq \mathbb{R}^d$ of the same co-volume are BD equivalent. In particular, any two lattices are BDD equivalent.*

Definition 1.3.7. A tiling τ of $\mathbb{R}^d$ is called *periodic* if it has d linearly independent translation symmetries $v_1, \ldots, v_d \in \mathbb{R}^d$.

Claim 1.3.8. *Let τ be a periodic tiling of $\mathbb{R}^d$, then every Y_τ is BDD to $\mathbb{Z}^d$.*

Proof. Denote by $T_1, \ldots, T_n$ the prototiles of τ. Let $\Gamma_1 = span_{\mathbb{Z}}\{v_1, \ldots, v_d\}$ be the lattice of periods of τ, and let K be the parallelepiped fundamental domain of Γ_1 spanned by $v_1, \ldots, v_d$. Denote by $\mathbb{T}$ the torus that is obtained by identifying parallel faces of K, then we have a tiling of $\mathbb{T}$ by $T_1, \ldots, T_n$. So by cutting along the tiles, one gets another fundamental domain K_1 of Γ_1 which is a union of tiles from $T_1, \ldots, T_n$.

Let m be the number of tiles in K_1, and set $\alpha = \frac{1}{m}\sum_{T \in K_1} Vol(T)$, the average volume of a tile in K_1. We show that every Y_τ is BD to $\beta\mathbb{Z}^d$, where $\beta = \alpha^{1/d}$. Denote by $e_1, \ldots, e_d$ the standard basis of $\mathbb{R}^d$ and let $\Gamma_2 = span_{\mathbb{Z}}\{m(\beta e_1), \beta e_2 \ldots, \beta e_d\}$, and $K_2 = [0, m\beta) \times [0, \beta) \times \ldots \times [0, \beta)$ its standard fundamental domain. Both K_1 and K_2 tiles $\mathbb{R}^d$ by translations, and $Vol(K_1) = Vol(K_2) = m\alpha$, so we may use the Hall's Marriage Theorem like in the proof of Claim 1.3.6 and get a bijection $g : \Gamma_1 \to \Gamma_2$ such that $(\gamma + K_1) \cap (g(\gamma) + K_2) \neq \varnothing$ for every $\gamma \in \Gamma_1$. Note that for every $\gamma \in \Gamma_1$ there are m elements of Y_τ in $\gamma + K_1$, and for every $\delta \in \Gamma_2$ there are m elements of $\beta\mathbb{Z}^d$ in $\delta + K_2$. So for every $\gamma \in \Gamma_1$ pick some bijection $h_\gamma : Y_\tau \cap (\gamma + K_1) \to \beta\mathbb{Z}^d \cap (g(\gamma) + K_2)$, and define $f : Y_\tau \to \beta\mathbb{Z}^d$ by

$$f(y) = \Big\{ h_\gamma(y), \quad y \in \gamma + K_1 \ .$$

It is easy to verify that f is as required. □

1.4 Substitution Tilings

Let $\xi > 1$ and let $\mathcal{F} = \{T_1, \ldots, T_k\}$ be a set of d-dimensional tiles.

Definition 1.4.1. A *substitution* is a mapping $H : \mathcal{F} \to \xi^{-1}\mathcal{F}^*$ such that $supp(T_i) = supp(H(T_i))$ for every i. Namely, it is a set of dissection rules that shows us how to divide the tiles to other tiles from $\mathcal{F}$ with a smaller scale. We also allow to apply H to finite or infinite collections of tiles. The constant ξ is called the *inflation constant* of H.

Definition 1.4.2. Let H be a substitution defined on $\mathcal{F}$. Consider the following set of patches:

$$\mathcal{P} = \{(\xi H)^m(T) : m \in \mathbb{N} \, , \, T \in \mathcal{F}\} \, .$$

The *substitution tiling space* X_H is the set of all tilings of $\mathbb{R}^d$ that for every patch P in them there is a patch $P' \in \mathcal{P}$ such that P is a sub-patch of P'. Every tiling $\tau \in X_H$ is called a *substitution tiling* of H.

There is a simple way to explicitly construct substitution tilings from a substitution rule H. Take one of the tiles T_i and apply ξH on it again and again, producing tilings of larger and larger regions. Let P_n be the tiling that is obtained after n iterations. So in all of these patches P_n there are only finitely many ways in which two tiles are glued together. Then if we fix some tile to be at the origin in all of these patches, for every $m \in \mathbb{N}$ the set $\{B(0, m) \cap P_n : n \in \mathbb{N}\}$ is finite. We have tilings of larger and larger regions, and there are only finitely many ways to tile each ball $B(0, m)$, so a standard compactness argument, like the König Lemma for example (see [K36]), gives the desired tiling.

One good reason to consider substitution tilings is that this is a way to construct non-periodic tilings, with finitely many tiles. This fact makes the study of the BD, BDD, and biLipschitz equivalence classes interesting in this context. A well known example for such a tiling is the Penrose Tiling of the plane. We present the substitution rule for one of its representations in the picture below. This tiling is by itself a reason to study substitution tilings, and indeed Burago and Kleiner asked in [BK02]: Is the set of vertices of a Penrose Tiling biLipschitz to $\mathbb{Z}^2$?

This question was answered positively in [S08], and a sketch of the same result also appears in [DSS95]. In [S11a], this result was extended, where it was proven that Y_τ is biLipschitz to $\mathbb{Z}^2$, for any primitive substitution tiling τ in $\mathbb{R}^2$. The proof in [S11a] relies on the following theorem of Burago and Kleiner, that gave a sufficient condition for a separated net to be biLipschitz to $\mathbb{Z}^2$.

Theorem 1.4.3 ([BK02], Theorem 1.3)**.** *Let Y be a separated net in $\mathbb{R}^2$. For a real number $\alpha > 0$ and a square Q with integer coordinates define:*

$$\begin{aligned} e_\alpha(Q) &= \max\left\{\frac{\alpha \cdot Vol(Q)}{\#(Q\cap Y)}, \frac{\#(Q\cap Y)}{\alpha \cdot Vol(Q)}\right\} \\ E_\alpha(2^i) &= \sup\left\{e_\alpha(Q) : Q \quad \textit{as above with an edge of length } 2^i\right\}. \end{aligned}$$

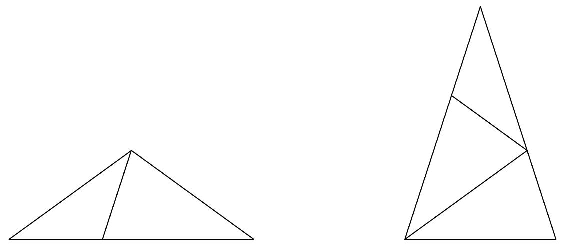
If there exists an $\alpha > 0$ such that the product $\prod_{j=1}^{\infty} E_\alpha(2^j)$ converges, then Y is biLipschitz to $\mathbb{Z}^2$.

In [ACG11], Aliste-Prieto, Coronel, and Gambaudo proved the d dimensional analog of this theorem, for any $d > 1$. As a corollary, they deduce that separated nets that correspond to primitive substitution tilings are biLipschitz equivalent to $\mathbb{Z}^d$ in any dimension d. So we move to discuss the BD and BDD equivalence relations, in the context of substitution tilings, and for that we need some more definitions.

Consider the following equivalence relation on tiles: $T_i \sim T_j$ if there exists an isometry O such that $T_i = O(T_j)$ and $H(T_i) = O(H(T_j))$. We call the representatives of the equivalence classes *basic tiles*, and denote them by $\{\mathcal{T}_1, \ldots, \mathcal{T}_n\}$. By this definition, we can also think of H as a dissection rule on the basic tiles and extend it to collections of tiles as before. For a tile T in the tiling we say that T is *of type i* if it is isometric to $\mathcal{T}_i$.

Definition 1.4.4. Let $\mathcal{F} = \{\mathcal{T}_1, \ldots, \mathcal{T}_n\}$ be the set of basic tiles. Define the *substitution matrix* of H to be an $n \times n$ matrix, $A_H = (a_{ij})$, where a_{ij} is the number of basic tiles in $\xi H(\mathcal{T}_j)$ which are of type i. We say that H is *primitive* if the matrix A_H is a primitive matrix. That is, if there exists an $m \in \mathbb{N}$ such that $A_H^m > 0$.

For example, if H is the substitution rule of the Penrose Tiling, as in the picture below, then $A_H = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.



The substitution matrix is a fundamental tool in the study of substitution tilings, and in all of the results that follow we always assume that it is primitive. Notice that the primitivity assumption is natural, since we expect to see all the tiles $T_1, \ldots, T_n$ from some point onwards, when applying H again and again on some basic tile T_i. If it does not happen, we will have tilings consist of only part of the tiles $\{T_1, \ldots, T_n\}$ in X_H.

1.5 Notations and Previous Results

A substitution tiling has many parameters that we need throughout the proofs in all three chapters. For the convenience of the reader we assemble all the notations regarding the parameters of the tiling here.

Our given tiling is usually denoted by τ or τ_0, and we fix a separated net $Y = Y_\tau$ that correspond to τ. The basic tiles are $\mathcal{F} = \{\mathcal{T}_1, \ldots, \mathcal{T}_n\}$, and $s_1, \ldots, s_n$ denotes their d-dimensional volume. H is the substitution, which is always assumed to be primitive, and $\xi > 1$ is the inflation constant. We denote by A_H the substitution matrix, and by $\eta_1, \ldots, \eta_n$ the eigenvalues of A_H in a descending order in absolute value, that is $\eta_1 > |\eta_2| \geq \ldots \geq |\eta_n|$. It is easy to check that $\eta_1 = \xi^d > 1$, and it follows from the Perron Frobenius Theorem that η_1 is of multiplicity one, and it has positive eigenvector v_1. We refer to either [BP79], [Gan59] or [H07] for further reading on the Perron Frobenius Theorem, and to §3 in [S11a] for more details on this theorem in the context of substitution tilings. We fix a Jordan basis of A_H and denote by v_i the i'th vector in it, where v_i corresponds to η_i, and we denote by $v(j)$ the j'th coordinate of the vector v. Without loss of generality, we normalize v_1 so that $v_1(1) = 1$. Denote by $u_1 = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}$, then it is easy to see that u_1 is the left eigenvector of A_H that corresponds to η_1.

The following proposition plays an important role in all of the three following chapters (it also appears in [Ro04]).

Proposition 1.5.1. *If H is a primitive substitution then* *(i)* $X_H \neq \emptyset$.

(ii) For every $\tau \in X_H$ *and for every* $m \in \mathbb{N}$ *there exists a tiling* $\tau_m \in X_H$ *that satisfies* $H^m(\tau_m) = \tau$.

Proof. (i) We already saw it, right after Definition 1.4.2.

(ii) It suffices to prove the claim for $m = 1$. We are looking for a tiling $\eta \in X_H$ such that $H\eta = \tau$. We build η as a limit of tilings $\eta_k \in \xi X_H$, where the standard topology on the tiling space is given by the metric $d(\eta_1, \eta_2) \leq 1/k$ if η_1, η_2 agree on the ball of radius k around 0. Since $\tau \in X_H$, every patch in τ is a sub-patch of $(\xi H)^m(T_i)$ for some m and i. Denote by B_k the patch that is obtained when intersecting τ with $B(0,k)$, the ball of radius k around 0. Let i_k, m_k be such that B_k is a sub-patch of $(\xi H)^{m_k}(T_{i_k})$. By primitivity, T_{i_k} appears in $\xi H(T_{j_k})$, for some tile T_{j_k}. So $(\xi H)^{m_k}(T_{i_k})$ is a sub-patch of $(\xi H)^{m_k+1}(T_{j_k})$, and therefore also B_k.

From the proof of (i) we may also deduce that for every patch of the form $P = (\xi H)^m T_i$ there is a tiling $\zeta \in X_H$ so that P is a sub-patch of ζ. Hence $P_k = \xi(\xi H)^{m_k}(T_{j_k})$ is a patch in some tiling $\eta'_k \in \xi X_H$, and moreover, $H(P_k)$ contains B_k as a sub-patch of it. So there are also tilings $\eta_k \in \xi X_H$ having P_k as a sub-patch of them, in way that $H(\eta_k) \cap B(0,k) = B_k$. So by the definition of the topology on X_H we have $\eta_k \to \eta$.

□

Given a tiling $\tau = \tau_0 \in X_H$, for every m we fix a tiling τ_m as in Proposition 1.5.1.

1.6 The Main Results of this Thesis

In this final section of the introduction we give a short promo for each one of the three following chapters. We present the questions we will be dealing with, and survey the related works on each of these questions. Some of the results of this thesis are contained in several papers. They are based on [S11a], published in the Israel Journal of Mathematics, which is the outcome of the author's MSC thesis. They contain the results of [S11b], that was published in the Proceedings of the American Mathematical Society, and of [S12], that was submitted to the Journal of Mathematical Analysis and Applications. Additionally, they contain results which will be submitted for publication in the near future.

In Chapter 2 we discuss the results from [S11b]. We deal with a property of tilings, with an analytic nature, that in particular implies that the corresponding separated net is biLipschitz to a lattice.

Definition 1.6.1. Let τ be a tiling of $\mathbb{R}^d$. Define a function $f_\tau : \mathbb{R}^d \to \mathbb{R}$ by

$$f_\tau(x) = \begin{cases} \frac{1}{Vol(T)} & \text{, there exists a tile } T \text{ in } \tau \text{ such that } x \in int(T) \\ 0 & \text{, otherwise} \end{cases},$$

where $int(T)$ is the interior of T.

Given a tiling τ of $\mathbb{R}^d$, we are interested in the question whether there is a biLipschitz homeomorphism $\varphi : \mathbb{R}^d \to \mathbb{R}$ with $Jac(\varphi)(x) = f_\tau(x)$, for almost every $x \in \mathbb{R}^d$, where $Jac(\varphi)(x) = D_\varphi(x)$. As shown in both [BK02] and [McM98], the existence of such a φ implies that Y_τ is biLipschitz to $\mathbb{Z}^d$. The proof goes simply by observing that $\{\varphi(T) : T \in \tau\}$ is a tiling of $\mathbb{R}^d$ by tiles of the same volume, and then applying Claim 1.3.5. This property stands in the heart of the study of the biLipschitz classes of separated nets. Theorem 1.4.3 is actually a corollary of another theorem regarding this property, and it was also used in [BK98] and [McM98]. In addition to what have been said, the question whether some target function can be realized as a Jacobian of another function with a given regularity has its own interest, and several variants of it were studied, see for instance [DM90] and [RY96]. Answering a question we were asked privately by McMullen, in Chapter 2 we focus on the question whether f_τ can be realized as a Jacobian, for a substitution tiling τ.

The question whether separated nets that comes from substitution tilings are BDD to a lattice or not was studied in [S12]. We devote Chapter 3 to discuss this question. It turns out that not all such separated nets are BDD to a lattice, and that the answer depends on the eigenvalues of the substitution matrix A_H. This question was studied earlier in [S11a] and [ACG11]. To describe the results, let $\eta_1 > |\eta_2| \geq \ldots, \geq |\eta_n|$ be the eigenvalues of A_H in a descending order. It was shown in [S11a] that the resulting separated net is BDD to a lattice wherever η_1 is a Pisot number. Equivalentely, if $|\eta_2| < 1$. Motivated by this initial result on that question, Aliste-Prieto, Coronel, and Gambaudo proved an improvement of that result in [ACG11], showing that the same hold if $|\eta_2| < \eta_1^{1/d}$ ($\eta_1^{1/d} = \xi > 1$). Our current result generalizes both of the previous ones, and answers the question whether the corresponding net is BDD to a lattice or not for almost all cases. Additionally, this work have promoted the interest in the BDD equivalence relation on separated nets, and indeed results of similar nature appear in [HKW12], where Haynes, Kelly, and Weiss study the biLipschitz and BDD equivalence classes of "cut-and-project nets".

In Chapter 4 we discuss the Danzer problem. This problem appears in the literature in several places, but even though it is open from the sixties, and can be asked without any prior knowledge, there is only one partial result that we could find on this problem. A set $D \subseteq \mathbb{R}^d$ is called a Danzer set if it intersect every convex set of volume 1. The original question of Danzer can be phrased as follows:

Question 1.6.2. *Is there a Danzer set $D \subseteq \mathbb{R}^d, (d > 1)$ with growth rate $O(T^d)$?*

The only partial result on this question that we know of belongs to Bambah and Wood, in [BW71], where they prove one positive and one negative result. On one hand, they show that there exists a Danzer set of growth rate $O(T^d(\log T)^{d-1})$, and on the other hand they show that a finite union of grids, i.e. translated lattices, is not a Danzer set. Our results here are of similar nature. We show that the Danzer problem is equivalent to a well studied question in combinatorics, and deduce the existence of a Danzer set of growth rate $O(T^d \log T)$, and then we show that nets that comes from substitution tilings are not Danzer sets.

Chapter 2

Functions of Substitution as a Jacobian

2.1 Introduction

This chapter has appeared as [S12], that was published in the Proceedings of the American Mathematical Society.

We recall the following definition from §1:

Definition 2.1.1. Let τ be a tiling of $\mathbb{R}^d$. Define a function $f_\tau : \mathbb{R}^d \to \mathbb{R}$ by

$$f_\tau(x) = \begin{cases} \frac{1}{Vol(T)}, & \text{there exists a tile } T \text{ in } \tau \text{ such that } x \in int(T) \\ 0, & \text{otherwise} \end{cases},$$

where $int(T)$ is the interior of T.

This chapter deals with the question whether the function f_τ, of a given tiling of $\mathbb{R}^d$, can be realized as the Jacobian of a biLipschitz homeomorphism of $\mathbb{R}^d$. As written in §1, a positive answer to this question implies that any separated net Y_τ is biLipschitz equivalent to $\mathbb{Z}^d$.

In this work we will not consider general functions f but rather only functions which, via Definition 2.1.1, come from substitution tilings of $\mathbb{R}^d$. To motivate this, recall that every tiling τ of $\mathbb{R}^d$ gives rise to separated nets Y_τ, simply by placing a point in each tile. These nets are in the same BD equivalence class, and in particular biLipschitz equivalent. We saw in §1.3 that it is enough to consider tilings with finitely many prototiles for the study of these equivalence relations. The results in [BK98] and [McM98] shows that not all f_τ can be realized as a Jacobian, and then deduce the existence of a separated net which is not biLipschitz to $\mathbb{Z}^d$. Combining the two last statements together implies that even for tilings τ with finitely many prototiles, f_τ might not be the Jacobian of any biLipschitz homeomorphism.

In [BK02] Burago and Kleiner gave a sufficient condition for a separated net to be biLipschitz equivalent to $\mathbb{Z}^2$, see Theorem 1.4.3. In fact, this theorem is a corollary of another theorem saying when a function $f : \mathbb{R}^2 \to (0, \infty)$, which is constant on unit lattice squares and with $\inf f > 0$, can be realized as a Jacobian of a biLipschitz homeomorphism of the plane. Their proof uses a sequence of partitions of the plane to larger and larger dyadic squares. The main idea of our proof is to look at this partition to dyadic squares as a special case of a substitution tiling of the space. That allows us to extend the theorem of Burago and Kleiner to the settings of substitution tilings, in any dimension, instead of dyadic squares decomposition in $\mathbb{R}^2$. Similarly to the result in [BK02], in our main theorem, Theorem 2.1.3, we show that f_τ of a primitive substitution tiling τ can be realized as a Jacobian if a local condition on the basic tiles is satisfied.

The second ingredient of our proof is a property of primitive substitution tilings that was obtained in [S11a], see Proposition 2.2.1. Using this proposition we can skip one of the main steps of the proof of Burago and Kleiner. For our more general settings the assumption in Theorem 2.1.3 is parallel to what Burago and Kleiner prove in their Proposition 3.2 for the case of dyadic squares in the plane.

In the main results that are stated below we use basic terminology from the theory of substitution tiling. We refer the reader to §1.4 and §1.5 for the relevant definitions and notations. In the context of substitution tiling, every tile T has a natural partition to smaller tiles, induced by the partitions of the substitution rule on the finite collection of basic tiles. For the statement of our main theorem, we add the following definition.

Definition 2.1.2. We say that a function $f : T \to \mathbb{R}$ is a *weight function* if it is constant and positive on the interiors of the tiles in the partition of T.

Theorem 2.1.3. *Let τ be a primitive substitution tiling of $\mathbb{R}^d$. Suppose that there is a constant C with the following property: for every basic tile $\mathcal{T}$, and for every weight function $f : \mathcal{T} \to (0, \infty)$, there is a biLipschitz homeomorphism $\varphi : \mathcal{T} \to \mathcal{T}$ with*

$$\varphi|_{\partial\mathcal{T}} = id, \quad Jac(\varphi) = \frac{\lambda(\mathcal{T})}{\int_{\mathcal{T}} f} \cdot f \quad a.e. \quad and \quad biLip(\varphi) \leq \left(\frac{\max f}{\min f}\right)^C. \tag{2.1}$$

Then there exists a biLipschitz homeomorphism $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ with $Jac(\Phi) = f_\tau$ a.e.

Remark 2.1.4. *The assumption in the theorem is stronger than what actually needed. In the proof we only use the assumption for weight functions with values* *which are averages of f_τ on tiles of the τ_m's. In particular, we only use it for bounded functions, $f(x) \in [1/L, L]$, where $L > 0$ depends on τ.*

As an application of the main result we give the following theorem on substitution tilings with star-shaped tiles. We elaborate on these objects in 2.4.1.

Theorem 2.1.5. *For any primitive star-shaped substitution tiling τ of $\mathbb{R}^2$, there is a biLipschitz homeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $Jac(\Phi) = f_\tau$ a.e.*

Corollary 2.1.6. *For any Penrose tiling τ there is a biLipschitz homeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $Jac(\phi) = f_\tau$ a.e.*

Notice that in the private case where we tile $\mathbb{R}^d$ by lattice cubes (each cube is divided to 2^d cubes), f_τ is a constant function. However, one may ask when a weight function which is constant on lattice cubes is a Jacobian? This question was answered in [ACG11], where they extend the main result of [BK02] to higher dimensions.

2.2 Preliminaries

We use the same definitions and notations for substitution tilings as in 1.5. We denote by $\lambda(T)$ the Lebesgue measure of a set $T \subseteq \mathbb{R}^d$, and $\#F$ for the number of elements in a finite set F. We use the separated net $Y = Y_\tau$, that corresponds to a tiling τ, to count the number of tiles in a patch P, since $\#(supp(P) \cap Y_\tau) = \#\{T \in \tau : T \in P\}$. Abusing notations, we denote this quantity by $\#(P \cap Y_\tau)$.

The next lemma is taken from [S11a] and give an estimate for the discrepancy of a tile $T \in \tau_m$, with respect to Y_τ.

Lemma 2.2.1. *Let τ be a primitive substitution tiling of $\mathbb{R}^d$, then there are positive constants $\kappa < \xi^d, c, \alpha$, that depends only on τ, such that for every m and $T \in \tau_m$, we have*

$$|\#(T \cap Y) - \alpha \cdot \lambda(T)| \leq c \cdot \kappa^m$$

Proof. See [S11a], Lemma 4.3. □

Note that if $T \in \tau_m$ is of type i then $\lambda(T) = (\xi^d)^m s_i$, where $s_i = \lambda(\mathcal{T}_i)$. Then we have the following immediate corollary.

Corollary 2.2.2. *Let τ be a primitive substitution tiling of $\mathbb{R}^d$, then there are positive constants $\zeta < 1, c, \alpha$, that depends only on τ, such that for every m and $T \in \tau_m$, we have*

$$\max\left\{\frac{|\#(T \cap Y) - \alpha \cdot \lambda(T)|}{\alpha \cdot \lambda(T)}, \frac{|\#(T \cap Y) - \alpha \cdot \lambda(T)|}{\#(T \cap Y)}\right\} \leq c \cdot \zeta^m. \tag{2.2}$$

The number α is the *asymptotic density of* Y, and we present and use it more explicitly in §3.

Throughout the whole chapter, we denote by $D_\phi(x)$ the derivative of $\phi : \mathbb{R}^d \to \mathbb{R}^d$ at the point x, and we use $Jac(\phi)$ to denote the Jacobian of ϕ, $Jac(\phi)(x) = \det(D_\phi(x))$.

2.3 Proof of the Main Theorem

In this section we prove Theorem 2.1.3. We follow the proof of Burago and Kleiner, modifying it to the context of substitution tilings in $\mathbb{R}^d$.

Proof of Theorem 2.1.3. First note that it is enough to find a biLipschitz homeomorphism Φ' with $Jac(\Phi') = \beta \cdot f_\tau$ for some positive constant β. Then $\Phi = \beta^{-1/d} \cdot \Phi'$ is as required.

For every $m \geq 1$ every tile $T \in \tau_m$ is tiled with tiles of τ_{m-1}. Define $f_m^T : T \to \mathbb{R}$ to be the average of f_τ on T', on every $T' \in \tau_{m-1} \cap T$:

$$f_m^T(x) = \begin{cases} \frac{\int_{T'} f_\tau}{\lambda(T')}, & x \in int(T'),\ T' \subseteq T,\ T' \in \tau_{m-1} \\ 0, & \text{otherwise} \end{cases}$$

Obviously f_m^T is a weight function (see Definition 2.1.2), then by the assumption, for every m and $T \in \tau_m$ there exists a biLipschitz homeomorphism $\varphi_m^T : T \to T$ that satisfies (2.1). Gluing these homeomorphisms along the boundaries of the tiles of τ_m gives a biLipschitz homeomorphism $\varphi_m : \mathbb{R}^d \to \mathbb{R}^d$ that satisfies

$$\begin{aligned} &Jac(\varphi_m)(x) \overset{a.e.}{=} \left\{ \frac{\lambda(T)}{\int_T f_m} \cdot f_m(x), \quad x \in int(T),\ T \in \tau_m \right. \\ &biLip(\varphi_m) \leq \left(\frac{\max f_m}{\min f_m} \right)^C, \end{aligned} \tag{2.3}$$

where $f_m : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$f_m(x) = \begin{cases} \frac{\int_{T'} f_\tau}{\lambda(T')}, & x \in int(T'),\ T' \in \tau_{m-1} \\ 0, & \text{otherwise} \end{cases}.$$

Define $\phi_n : \mathbb{R}^d \to \mathbb{R}^d$ by $\phi_n = \varphi_n \circ \varphi_{n-1} \circ \ldots \circ \varphi_1$. We claim that (ϕ_n) has a subsequence that converges uniformly to the desired Φ.

First note that for any patch P we have

$$\int_P f_\tau = \#\{S \in \tau_0 : S \subseteq P\} = \#(P \cap Y).$$

Combining this with (2.2) we obtain

$$\begin{cases} \alpha^{-1}\max f_m \le \max\limits_{T\in\tau_{m-1}} \left\{\frac{\#(T\cap Y)}{\alpha\lambda(T)}\right\} \le 1+c\zeta^m \\ \frac{\alpha}{\min f_m} \le \max\limits_{T\in\tau_{m-1}} \left\{\frac{\alpha\lambda(T)}{\#(T\cap Y)}\right\} \le 1+c\zeta^m \end{cases} \implies \frac{\max f_m}{\min f_m} \le (1+c\zeta^m)^2.$$

Then for every n we have

$$biLip(\phi_n) \le \prod_{m=1}^{n} \left(\frac{\max f_m}{\min f_m}\right)^C \le \left(\prod_{m=1}^{\infty} 1 + c\cdot\zeta^m\right)^{2C}.$$

Since

$$\log\left(\prod_{m=1}^{\infty} 1+c\cdot\zeta^m\right) \le \sum_{m=1}^{\infty} \log\left(1+c\cdot\zeta^m\right) \le \sum_{m=1}^{\infty} (c\cdot\zeta^m) = \frac{c\zeta}{1-\zeta},$$

we have a uniform bound for $biLip(\phi_n)$. Then by the Arzela Ascoli Theorem we get a biLipschitz homeomorphism $\Phi : \mathbb{R}^d \to \mathbb{R}^d$, with $biLip(\Phi) \le e^{2C\cdot c\cdot\zeta/(1-\zeta)}$.

Regarding $Jac(\Phi)$, for any $T \in \tau_m$

$$\int_T f_m = \sum_{\substack{T'\subseteq T\\ T'\in\tau_{m-1}}} \frac{1}{\lambda(T')}\int_{T'} \#(T'\cap Y) = \#(T\cap Y). \tag{2.4}$$

By (2.3), for every $T \in \tau_m, T' \in \tau_{m-1}$ and for a.e. $x \in T' \subseteq T$ we have

$$Jac(\varphi_m)(x) = \frac{\lambda(T)}{\int_T f_m}\cdot f_m(x) = \frac{\lambda(T)}{\#(T\cap Y)}\cdot\frac{\#(T'\cap Y)}{\lambda(T)}. \tag{2.5}$$

Take $x \in \mathbb{R}^d$, denote by T_m the tile of τ_m that contains x, and suppose that $x \in int(T_0)$. Since for every m the map φ_m maps every tile of τ_m to itself, and by (2.2), we have

$$Jac(\phi_n)(x) = Jac(\varphi_n)(\varphi_{n-1}\circ\ldots\circ\varphi_1(x))\cdot Jac(\varphi_{n-1})(\varphi_{n-2}\circ\ldots\circ\varphi_1(x))\cdot\ldots\cdot Jac(\varphi_1)(x) \overset{(2.5)}{=}$$

$$\frac{\lambda(T_n)\cdot\#(T_{n-1}\cap Y)}{\#(T_n\cap Y)\cdot\lambda(T_{n-1})}\cdot\ldots\cdot\frac{\lambda(T_1)\cdot\#(T_0\cap Y)}{\#(T_1\cap Y)\cdot\lambda(T_0)} = \frac{\lambda(T_n)}{\#(T_n\cap Y)}\cdot\frac{1}{\lambda(T_0)} \xrightarrow{n\to\infty} \frac{\alpha^{-1}}{\lambda(T_0)} = \alpha^{-1}f_\tau(x).$$

$\square$

2.4 The Star-Shaped Lemma

The purpose of this section is to obtain a homeomorphism between star-shaped domains in the plane, with Jacobian 1 a.e. This result will be used in the next section. We also develop the terminology that we need for dealing with star-shaped domains here.

Definition 2.4.1. $T \subseteq \mathbb{R}^d$ is a *star-shaped domain* if there exists a point $p \in int(T)$ such that for every point $x \in T$ the interval between p and x is contained in T, that is $\{tp + (1-t)x : t \in [0,1]\} \subseteq T$. We denote $\langle T, p \rangle$ a star-shaped domain T with a point p as above. For short, we say that p *sees all of* T for this property. Given a star-shaped domain $\langle T, p \rangle$ in $\mathbb{R}^2$, and assume that $p = 0$, every $\theta \in [0, 2\pi]$ defines a *θ-sector of* T in polar coordinates by:

$$T^{(\theta)} = \{(r, \alpha) \in T : \alpha \in [0, \theta]\}.$$

Lemma 2.4.2. *Suppose $\langle T_1, p_1 \rangle, \langle T_2, p_2 \rangle$ are two star-shaped domains in the plane, with piecewise differentiable boundary and with the same area, then there is a unique homeomorphism $\psi : T_1 \to T_2$, such that:*

- *$\psi(p_1) = p_2$.*
- *ψ maps ∂T_1 injectively onto ∂T_2.*
- *ψ maps every sector of T_1 to a sector of T_2 with the same area.*
- *$Jac(\psi) = 1$ a.e.*

Proof. It suffices to show that there is such a homeomorphism between the unit ball $B = B(0,1)$ and another star-shaped domain $\langle T, 0 \rangle$ with the same area.

Define a mapping $\psi : B \to T$ by

$$\psi(r\cos(\theta), r\sin(\theta)) = r \cdot R(\beta) \cdot (\cos(\beta), \sin(\beta)),$$

where $R(\kappa)$ is the distance between 0 and ∂T in direction $\kappa \in [0, 2\pi]$, and β is an angle that is defined by the following equation:

$$\frac{1}{2}\int_0^\beta R^2(t)dt = \frac{\theta}{2}.$$

Namely, for every θ we choose $\beta = \beta(\theta)$ such that the sector of angle β in T has the same area as the sector of angle θ in B.

It is now left to the reader to check that ψ satisfies the requirements. $\square$

2.5 Application for Star-Shaped Substitution Tilings

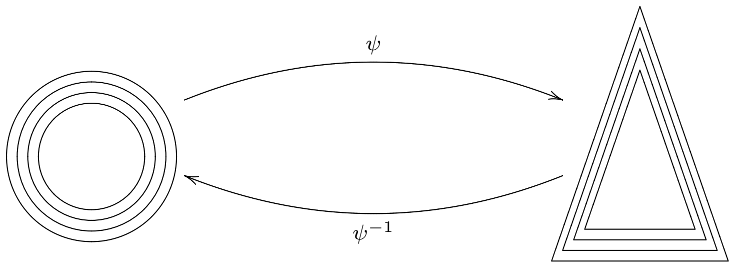
Definition 2.5.1. Let τ be a primitive substitution tiling $\mathbb{R}^d$. We say that τ is a *star-shaped substitution tiling* if every tile of τ is star-shaped with a piecewise differentiable boundary.

In this section we prove Proposition 2.5.2, that shows that the hypothesis of Theorem 2.1.3 is satisfied for star-shaped substitution tilings of the plane. Proposition 2.5.2 generalized Proposition 3.2 of [BK02] from dyadic lattice square tiling to any star-shaped substitution tiling. The proof is obtained by repeating the steps of their proof, with the proper modifications.

Proposition 2.5.2. *Let $\langle T, p\rangle$ be a star-shaped domain, with a partition to smaller star-shaped domains $\langle T_1, p_1\rangle, \ldots, \langle T_n, p_n\rangle$. Then there is a constant C_1 such that for every weight function $f : T \to (0, \infty)$ there is a biLipschitz homeomorphism $\varphi : T \to T$ that satisfies (2.1), with C_1 instead of C.*

For the proof of Proposition 2.5.2 we need the following definitions:

Definition 2.5.3. Let $\langle T, p\rangle$ be a star-shaped domain and assume that $\lambda(T) = \lambda(B)$, where $B = B(0, 1)$. Let $\psi : B \to T$ be the homeomorphism from Lemma 2.4.2. For a given function $f : T \to \mathbb{R}$, we say that f *is constant on the elevation lines of* T if f is constant on (one dimensional) sets of the form $\psi(\partial B(0, r))$. In a similar manner we can define objects like *contraction around* p, *star-shaped annulus*, *neighborhood of the boundary*, etc. We will also use the same terminology for $\psi : S \to T$, where S is a square instead of a ball.



With this terminology we can now state the two lemmas which play the role of Lemma 3.3 and Lemma 3.6 from [BK02].

Lemma 2.5.4. *Let $\langle T, p\rangle$ be a star-shaped domain. There is a constant C_2 with the following property: Suppose $h_1, h_2 : T \to \mathbb{R}$ are continuous positive functions, which are constant on the elevation lines of T, and $\int_T h_1 = \int_T h_2$. Then there exists a biLipschitz homeomorphism $\phi : T \to T$, which fixes ∂T pointwise, so that $Jac(\phi) = \frac{h_1}{h_2 \circ \phi}$ a.e. and*

$$biLip(\phi) \leq \left(\frac{\max h_1}{\min h_1}\right)^{C_2} \left(\frac{\max h_2}{\min h_2}\right)^{C_2}. \tag{2.6}$$

Lemma 2.5.5. *Let $\langle T, p\rangle$ be a star-shaped domain and let A be a star-shaped annulus that is obtained by removing a contracted copy of T around p. Then there is a constant C_3 such that for every $g_1, g_2 : A \to \mathbb{R}$, positive Lipschitz functions with*

$$\int_A g_1 = \int_A g_2 = \lambda(A),$$

there is a biLipschitz homeomorphism $\phi : A \to A$ with $Jac(\phi) = \frac{g_1}{g_2 \circ \phi}$ a.e., and

$$biLip(\phi) \leq \left[\frac{\max g_1}{\min g_1}(1 + Lip(g_1))\right]^{C_3} \cdot \left[\frac{\max g_2}{\min g_2}(1 + Lip(g_2))\right]^{C_3}.$$

Moreover, when $g_1|_{\partial A} = g_2|_{\partial A}$, then ϕ can be chosen to fix ∂A pointwise.

Proof of Lemma 2.5.4. As in [BK02], the general case follows from the special case $h_2 \equiv 1$, $\int_T h_1 = \lambda(T)$. Indeed, setting $\overline{h_i} = (\lambda(T)/\int_T h_i)h_i$ we have $\int_T \overline{h_i} = \lambda(T)$ for $i \in \{1, 2\}$. Applying the result of the special case we get biLipschitz homeomorphisms $\phi_1, \phi_2 : T \to T$ with $Jac(\phi_i) = \overline{h_i}$, and

$$biLip(\phi_i) \leq \left(\frac{\max \overline{h_i}}{\min \overline{h_i}}\right)^{C_2} = \left(\frac{\max h_i}{\min h_i}\right)^{C_2}.$$

Then $\phi = \phi_2^{-1} \circ \phi_1$ satisfies $Jac(\phi)(x) = Jac(\phi_2^{-1})(\phi_1(x)) \cdot Jac(\phi_1)(x) = \frac{\overline{h_1}(x)}{\overline{h_2}(\phi_2^{-1} \circ \phi_1(x))} = \frac{h_1}{h_2 \circ \phi}(x)$ a.e., and $biLip(\phi)$ satisfies (2.6).

For that special case, we denote $h := h_1$, and assume without loss of generality that $\lambda(T) = \lambda(S_1)$, where $S_r = B_\infty(0, r) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_\infty \leq r \right\}$, and $\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_\infty = \max\{|x|, |y|\}$. Let $\psi : S_1 \to T$ be as in Lemma 2.4.2. Denote by $f = h \circ \psi$, then $f : S_1 \to \mathbb{R}$ satisfies the conditions of Lemma 3.3 from [BK02]. Define $g : [0, 1] \to [0, 1]$, and a biLipschitz homeomorphism $\widetilde{\phi} : S_1 \to S_1$, by

$$g(r) = \frac{1}{2}\sqrt{\int_{S_r} f}, \qquad \text{and} \qquad \widetilde{\phi}(x) = g(\|x\|_\infty) \cdot \frac{x}{\|x\|_\infty}.$$

It was proved in [BK02, proof of Lemma 3.3] that

$$Jac(\widetilde{\phi}) = f \text{ a.e. }, \quad biLip(\widetilde{\phi}) \leq k_1 \frac{\max f}{\min f} = k_1 \frac{\max h}{\min h}, \tag{2.7}$$

and, when $\frac{\max h}{\min h}$ is close to 1

$$\left\| D_{\widetilde{\phi}} - I \right\| \leq k_2 \left(\frac{\max f}{\min f} - 1\right) = k_2 \left(\frac{\max h}{\min h} - 1\right), \tag{2.8}$$

where k_1 and k_2 are independent of h.

Define $\phi = \psi \circ \widetilde{\phi} \circ \psi^{-1} : T \to T$. Then, by the chain rule, ϕ is a biLipschitz homeomorphism that satisfies:

$$Jac(\phi)(x) = \underbrace{Jac(\psi)(\widetilde{\phi} \circ \psi^{-1}(x))}_{=1} \cdot \underbrace{Jac(\widetilde{\phi})(\psi^{-1}(x))}_{=f(\psi^{-1}(x))} \cdot \underbrace{Jac(\psi^{-1})(x)}_{=1} = h(x) \text{ a.e.}$$

By (2.7) and (2.8), there are C, C', that depend on ψ, such that

$$biLip(\phi) \leq biLip(\psi) \cdot biLip(\widetilde{\phi}) \cdot biLip(\psi^{-1}) \leq k_1 \cdot C \cdot \frac{\max f}{\min f} = k_1' \cdot \frac{\max h}{\min h}, \quad (2.9)$$

and when $\frac{\max h}{\min h}$ is close to 1,

$$\|(D_\phi - I)(x)\| = \left\| D_{\psi \circ (\widetilde{\phi} - I) \circ \psi^{-1}}(x) \right\| \leq C' \cdot \left\| (D_{\widetilde{\phi}} - I)(\psi^{-1}(x)) \right\| = k_2' \left(\frac{\max h}{\min h} - 1 \right),$$

where k_1' and k_2' do not depend on h. This implies that

$$\left\| D_\phi^{\pm 1} \right\| \leq \left(\frac{\max h}{\min h} \right)^{k_2''}, \quad (2.10)$$

where k_2'' does not depend on h. Combining (2.9) and (2.10) we get

$$biLip(\phi) \leq \left(\frac{\max h}{\min h} \right)^{C_2},$$

as required. □

The proof of Lemma 2.5.5 is obtained in a similar way by following the proof of Lemma 3.6 of [BK02], and using ψ from Lemma 2.4.2 as we did in the proof of Lemma 2.5.4.

Finally, before we approach the proof of Proposition 2.5.2 we need the following claim:

Claim 2.5.6. *Let $\langle T, p \rangle$ be a star-shaped domain, with a partition to smaller star-shaped domains $\langle T_1, p_1 \rangle, \ldots, \langle T_n, p_n \rangle$. For an $r \in (0,1)$ we denote by T_i^r the contraction of T_i by r around p_i. Then there exists an $r > 0$ and a point $q \in int(T) \setminus \bigcup T_i^r$ such that for every $i \in \{1, \ldots, n\}$ and $x \in T_i^r$, the interval between q and x is contained in T.*

Proof. We say that "*q sees y*" if $\{tq + (1-t)y : t \in [0,1]\} \subseteq T$, and denote

$$T_{1-\varepsilon} = \{x \in T : d(x, \partial T) \geq \varepsilon\}.$$

We first show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $q \in B(p, \delta)$ and $y \in T_{1-\varepsilon}$, q sees y. Assume otherwise, given an $\varepsilon > 0$, for every $n \in \mathbb{N}$ there is a $q_n \in B(p, \frac{1}{n})$ and a point $y_n \in T_{1-\varepsilon}$ such that q_n does not see y_n. That is, for every n there is a $t_n \in [0,1]$ such that $z_n = t_n q_n + (1-t_n) y_n \notin T$.

We know that $q_n \xrightarrow{n\to\infty} p$, and by passing to subsequences we may assume that $t_n \xrightarrow{n\to\infty} t \in [0,1]$ and $y_n \xrightarrow{n\to\infty} y \in T_{1-\varepsilon}$. Then $z_n \xrightarrow{n\to\infty} z = tp + (1-t)y \notin int(T)$. We have obtained a point $y \in T_{1-\varepsilon}$, in particular $y \in int(T)$, such that the interval between p and y travels out of $int(T)$, a contradiction.

Since $p_1, \ldots, p_n \in int(T)$, we may fix $r_1, \varepsilon > 0$ such that for all $0 < r < r_1$ we have $\bigcup T_i^r \subseteq T_{1-\varepsilon}$. Let $\delta > 0$ be as above, so there is an $r_2 > 0$ such that the union $\bigcup T_i^{r_2}$ does not cover all of $B(p,\delta)$. So for $r = \min\{r_1, r_2\}$, any point $q \in B(p,\delta) \smallsetminus \bigcup T_i^r$ is as required. □

We are ready to prove Proposition 2.5.2. The proof follows the steps of the proof of Proposition 3.2 in [BK02], replacing Lemmas 3.3 and 3.6 there by Lemmas 2.5.4 and 2.5.5 from above. For the convenience of the reader we repeat their proof, for the more general context of substitution tilings.

Proof of proposition 2.5.2. Let $r > 0$ and let $q \in int(T) \smallsetminus \bigcup T_i^r$ be as in Claim 2.5.6. Define

$$S = \{y \in T : q \text{ sees } y\},$$

and let A be the star-shaped annulus that is obtained by removing from S a contracted copy of S around q, S', such that $S \smallsetminus S'$ still contains $\bigcup T_i^r$.

We may assume that $\int_T f = \lambda(T)$. Define the following functions:
Let $f_2 : T \to (0,\infty)$ be a Lipschitz function such that $f_2 = \min f$ on the complement of $\bigcup T_i^r$, and is constant on the elevation lines of each of the T_i's. In addition

$$\int_{T_i} f_2 = \int_{T_i} f, \quad \frac{\max f_2}{\min f_2} \le \left(\frac{\max f}{\min f}\right)^{k_1}, \quad \text{and} \quad 1 + Lip(f_2) \le \left(\frac{\max f}{\min f}\right)^{k_1},$$

where k_1 is independent of f.
Let $f_3 : T \to (0,\infty)$ be a Lipschitz function such that $f_3 = \min f$ on the complement of A, f_3 is constant on the elevation lines of S, and

$$\int_{T} f_3 = \int_{T} f, \quad \frac{\max f_3}{\min f_3} \le \left(\frac{\max f}{\min f}\right)^{k_2}, \quad \text{and} \quad 1 + Lip(f_3) \le \left(\frac{\max f}{\min f}\right)^{k_2},$$

where k_2 is independent of f.
Finally, set $f_4 = 1$.

Since for every $i \in \{1, \ldots, n\}$ we have $\int_{T_i} f_2 = \int_{T_i} f$, we can apply Lemma 2.5.4 on $f|_{T_i}$ and $f_2|_{T_i}$, separately for every i, and get a biLipschitz homeomorphism $\psi_1^i : T_i \to T_i$ with $Jac(\psi_1^i) = \frac{f}{f_2 \circ \psi_1^i}$ a.e. Gluing these homeomorphisms along the boundaries of the T_i's we obtain a biLipschitz homeomorphism $\psi_1 : T \to T$ with

$$Jac(\psi_1) = \frac{f}{f_2 \circ \psi_1} \text{ a.e.} \quad \text{and} \quad biLip(\psi_1) \le \left(\frac{\max f}{\min f}\right)^{C_2 k_1}.$$

Since for every $x \in T \setminus A$ we have $f_2(x) = f_3(x) = \min f$, $\int_A f_2 = \int_A f_3 \geq \lambda(A)$. Define $\bar{f}_j = \left(\lambda(A)/\int_A f_j\right) \cdot f_j$, for $j \in \{2,3\}$, so $Lip(\bar{f}_j) \leq Lip(f_j)$. Applying Lemma 2.5.5 to the star-shaped annulus A, with $\bar{f}_2$ and $\bar{f}_3$, we get a biLipschitz homeomorphism $\bar{\psi}_2 : A \to A$, that fixes ∂A pointwise, with

$$Jac(\bar{\psi}_2) = \frac{\bar{f}_2}{\bar{f}_3 \circ \bar{\psi}_2} = \frac{f_2}{f_3 \circ \bar{\psi}_2} \text{ a.e.} \qquad \text{and} \quad biLip(\bar{\psi}_2) \leq \left(\frac{\max f}{\min f}\right)^{C_3 k_2}. \tag{2.11}$$

We can extend $\bar{\psi}_2$ to $\psi_2 : T \to T$ by defining it to be the identity outside of A, and we get a biLipschitz homeomorphism of T, satisfying (2.11) (since outside of A we get $\frac{f_2}{f_3 \circ \psi_2}(x) = 1$).

Finally, we apply Lemma 2.5.4 again on f_3 and f_4, to get a biLipschitz homeomorphism $\psi_3 : T \to T$ with

$$Jac(\psi_3) = \frac{f_3}{f_4 \circ \psi_3} \text{ a.e.} \qquad \text{and} \quad biLip(\psi_3) \leq \left(\frac{\max f}{\min f}\right)^{C_2 k_2}.$$

Now define $\phi = \psi_3 \circ \psi_2 \circ \psi_1$. So $C_1 = C_2 k_1 + C_3 k_2 + C_2 k_2$ satisfies the statement of the proposition, and we have

$$Jac(\phi)(x) = Jac(\psi_3)(\psi_2 \circ \psi_1(x)) \cdot Jac(\psi_2)(\psi_1(x)) \cdot Jac(\psi_1)(x) =$$

$$\frac{f_3}{f_4 \circ \psi_3}(\psi_2 \circ \psi_1(x)) \cdot \frac{f_2}{f_3 \circ \psi_2}(\psi_1(x)) \cdot \frac{f}{f_2 \circ \psi_1}(x) = \frac{f}{f_4 \circ \phi}(x) = f(x)$$

as required. □

Theorem 2.1.5 is a direct consequence of Theorem 2.1.3 and Proposition 2.5.2. Corollary 2.1.6 follows directly from Theorem 2.1.5

Chapter 3

Bounded Displacement Equivalence on Separated Nets

3.1 Introduction

This chapter appears in [S11b], that was submitted to the Journal of Mathematical Analysis and Applications.

We recall the following definition from §1. Given two separated nets $Y_1, Y_2 \subseteq \mathbb{R}^d$, we say that Y_1 *is a bounded displacement (BD) of* Y_2 if there exists a bijection $\varphi : Y_1 \to Y_2$ with

$$\sup_{y \in Y_1} \{d(y, \varphi(y))\} < \infty.$$

Y_1 *is a bounded displacement after dilation (BDD) of* Y_2 if there is a constant $\alpha > 0$ such that Y_1 is BD to $\alpha \cdot Y_2$.

This chapter deals with the following question:

Question 3.1.1. *Given a separated net $Y \subseteq \mathbb{R}^d$, is Y BDD to a lattice?*

As mentioned in §1, there exists separated nets in $\mathbb{R}^d$ which are not biLipschitz equivalent to a lattice (for $d > 1$), and this in particular implies that there are separated nets which are not BDD to a lattice. But finding an example for a separated net which is not BDD to a lattice is a much simpler question, and one simple example for such a net is obtained by simply combining two lattices with different covolumes:

$$Y = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix} \in \mathbb{Z}^d : a_1 \geq 0 \right\} \cup \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix} \in (2\mathbb{Z})^d : a_1 < 0 \right\}.$$

To show explicitly that Y is not BD to any lattice, one shall use the Hall's Marriage Theorem, see Theorem 1.3.3. It is not difficult to show that for any $M, \alpha > 0$, when considering the bipartite graph $G = (\alpha\mathbb{Z}^d \cap Y, E)$, where $\{z, y\} \in E \overset{\text{iff}}{\iff} d(z, y) \leq M$, the Hall's condition is violated.

As we explained in §1.3, in the context of Question 3.1.1 it is equivalent to consider separated nets in $\mathbb{R}^d$ that corresponds to tilings with finitely many tiles, up to isometry. We restrict ourselves to substitution tilings, see §1.4, and focus on the following question:

Question 3.1.2. *Given a substitution tiling τ of $\mathbb{R}^d$, is Y_τ BDD to a lattice?*

These tilings are interesting because, on one hand, they tile by only finitely many tiles, and on the other hand, they are usually non periodic. The main theorem of this chapter, Theorem 3.1.3, answers Question 3.1.2 almost completely.

Substitution tilings has a corresponding matrix, the substitution matrix, which we denote by A_H, see Definition 1.4.4. We denote by $\eta_1, \dots, \eta_n$ the eigenvalues of A_H, with a descending order in absolute value. These parameters play an important role in our main results, and in the previous related results.

Question 3.1.2 was previously studied in [S11a] and [ACG11]. It was shown in [S11a] that any primitive substitution tiling with a matrix A_H of Pisot type, namely $|\eta_2| < 1$, gives rise to a separated net which is a BDD to a lattice. Recently Aliste-Prieto, Coronel and Gambaudo have improved this result. They showed that the same holds if $|\eta_2| < \eta_1^{1/d}$, see [ACG11]. We recall from §1.5 that $\eta_1 > 1$, so this is indeed an improvement. Our Theorem 3.1.3, extends the results of [S11a] and [ACG11] to a wider class of tilings, and gives the tight inequality on the eigenvalues of A_H, saying when a BDD to $\mathbb{Z}^d$ exists, and when it does not. These results were written in [S12].

Shortly after the results in [S12] came out, similar results were proved for another family of separated nets, which sometimes called *cut and project nets* or *quasicrystals*. The resulting net $Y \subset \mathbb{R}^d$ depends on the following parameters: two integers $2 \leq d \leq k$, a linear subspace $V \cong \mathbb{R}^d$, a point $x \in \mathbb{T}^k$, and a Poincar'e section $\mathcal{S} \subseteq \mathbb{T}^k$. We rather not to elaborate on these constructions here, and refer to [BM00], [HKW12], [M94], and [Se95]. In [HKW12], Haynes, Kelly and Weiss prove two main results. In their first result they show that almost every cut and project net is biLipschitz equivalent to a lattice. The second result deals with the BDD equivalence relation on separated nets. They find two mild conditions on the parameters for which the resulting net is BDD to a lattice. Then they present a diphantine condition on V, that gives a residual set of subspaces, for which the resulting net is not BDD to a lattice.

To state the main result, we denote by W_η the eigenspace that corresponds to η, by $W^\perp$ the subspace which is orthogonal to W with respect to the standard inner product $\langle \cdot, \cdot \rangle$, and set $\mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^d$.

Theorem 3.1.3. *For a primitive substitution tiling of $\mathbb{R}^d$, fix $t \geq 2$ to be the minimal index that satisfies $W_{\eta_t} \nsubseteq \mathbf{1}^{\perp}$. Then the corresponding separated net Y satisfies the following:*

(I) If $|\eta_t| > \eta_1^{\frac{d-1}{d}}$ then Y is not a BDD of $\mathbb{Z}^d$.

(II) If $|\eta_t| < \eta_1^{\frac{d-1}{d}}$ then Y is a BDD of $\mathbb{Z}^d$.

(III) If $|\eta_t| = \eta_1^{\frac{d-1}{d}}$ and η_t has a non-trivial Jordan block, then Y is not a BDD of $\mathbb{Z}^d$. Moreover, there are cases where the same consequence holds, and η_t has a trivial Jordan block.

Remark 3.1.4.

- *Note that $t = 2$ for almost every matrix A_H.*
- *It is follows from the proof Lemma 3.2.1 that if there is no t as above, namely $W_{\eta_t} \subseteq \mathbf{1}^{\perp}$ for every $t \neq 1$, then Y is a BDD to $\mathbb{Z}^d$.*
- *In the case of equality $|\eta_t| = \eta_1^{\frac{d-1}{d}}$, we do not know if there is an example in which Y is a BDD to $\mathbb{Z}^d$.*

The proof of the theorem relies on the following result of Laczkovich, that gives equivalent conditions for a discrete set in $\mathbb{R}^d$ to be a BDD to $\mathbb{Z}^d$.

Theorem 3.1.5 ([L92])**.** *For a discrete set $Y \subseteq \mathbb{R}^d$ and $\beta > 0$ the following statements are equivalent:*

(i) There is a constant C such that for any measurable set $A \subseteq \mathbb{R}^d$ we have

$$|\#(Y \cap A) - \beta \cdot \lambda(A)| \leq C \cdot \lambda\left(\{x \in \mathbb{R}^d : d(x, \partial A) \leq 1\}\right).$$

(ii) There is a constant C such that for every finite union of unit lattice cubes U we have

$$|\#(Y \cap U) - \beta \cdot \lambda(U)| \leq C \cdot \lambda_{d-1}(\partial U).$$

(iii) There is a BD $\phi : Y \to \beta^{-1/d}\mathbb{Z}^d$.

Where $\lambda_{d-1}(A)$ is the $d-1$-dimensional Lebesgue measure of A.

To prove Theorem 3.1.3 we show for each case that condition (i), or (ii), holds, or does not hold, and that way deduce whether (iii) holds or not. To get the estimates for the discrepancy in either (i) or (ii) we use an improved version of Lemma 4.3 from [S11a].

The organization of this chapter is as follows: In §3.2 we recall some definitions, gather all the relevant notations, and mention a few results on substitution tilings that will be used in the following sections. In §3.3 we get a series of estimates that are needed for the proof of Theorem 3.1.3. Among them, we prove an isoperimetric lemma, and then use it to generalize a result of Laczkovich to the context of substitution tilings. This result explain how to get any patch using unions and differences on tiles from different generations τ_m, and it might be of interest on its own. In §3.4 we prove Theorem 3.1.3, and finally we give examples for the different cases of this Theorem in §4.2.7.

3.2 Background

Throughout this chapter, a *tile* $T \subseteq \mathbb{R}^d$ is a set which is biLipschitz homeomorphic to a closed d-dimensional ball. Note that this requirement already implies that the tile's boundary has a well defined $d-1$-dimensional volume. We use all the definition and notations of substitution tilings from §1.4 and §1.5. In addition, we denote by e_i the i'th element of the standard basis of $\mathbb{R}^n$.

We use vectors to represent the number of basic tiles from each type in a given patch, e.g. e_i represents one tile of type i. Taking the substitution matrix A_H we get that $A_H \cdot e_i$ is the i'th column of A_H. Thus, by definition 1.4.4, it gives a vector that represents the number of basic tiles of each type obtained after applying H on $\mathcal{T}_i$. By linearity of A_H, this idea extends to any vector in $\mathbb{R}^n$.

Recall that our given tiling is denoted by τ or τ_0, and we fix some separated net $Y = Y_\tau$ that correspond to τ. The basic tiles are $\mathcal{F} = \{\mathcal{T}_1, \ldots, \mathcal{T}_n\}$, and $s_1, \ldots, s_n$ denotes their d-dimensional volume. H is the substitution rule, and $\xi > 1$ is the inflation constant. We denote by $\eta_1, \ldots, \eta_n$ the eigenvalues of A_H in a descending order in absolute value. η_1 is of multiplicity one, it satisfies $\eta_1 = \xi^d > 1$ and $\eta_1 > |\eta_i|$ for every i, and his eigenvector v_1 is positive (see §1.5). We fix a Jordan basis of A_H and denote by v_i the i'th vector in it, where v_i corresponds to η_i, and by $v(j)$ the j'th coordinate of the vector v. Without loss of generality, we normalize v_1 so that $v_1(1) = 1$. Denote by $u_1 = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}$, then it is easy to see that u_1 is the left eigenvector of A_H that corresponds to η_1. Finally, we fix

$$\alpha = \frac{\sum_{i=1}^n v_1(i)}{\sum_{i=1}^n v_1(i) \cdot s_i} = \frac{\langle \mathbf{1}, v_1 \rangle}{\langle u_1, v_1 \rangle}. \tag{3.1}$$

This α is the asymptotic density of Y.

Like in the previous two chapters, given a tiling $\tau = \tau_0 \in X_H$, for every $m \in \mathbb{N}$ we fix a tiling τ_m as in Proposition 1.5.1. For the proofs of this chapter, $\mathscr{T}^{(m)}$

denotes the set of all tiles of τ_m, and $\mathscr{T} = \bigcup_m \mathscr{T}^{(m)}$. The set of all finite unions of tiles of τ_0 is denoted by $\mathscr{V}$.

We prove Theorem 3.1.3 using Theorem 3.1.5. To use it we need to estimate the discrepancy $|\#(Y \cap U) - \alpha \cdot \lambda(U)|$ for different sets U. Notice that for every patch $V \in \mathscr{V}$ we have

$$\#(Y \cap V) = \sum_{i=1}^{n} a_i = \langle \mathbf{1}, a_V \rangle, \quad \text{and} \quad \lambda(V) = \sum_{i=1}^{n} a_i \cdot s_i = \langle u_1, a_V \rangle, \tag{3.2}$$

where $a_V = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, and a_j is the number of tiles of τ_0 from type j in V. Then the discrepancy of V depend only on a_V, and is given by the absolute value of the following linear functional:

$$disc(a_V) = \langle \mathbf{1}, a_V \rangle - \frac{\langle \mathbf{1}, v_1 \rangle}{\langle u_1, v_1 \rangle} \langle u_1, a_V \rangle. \tag{3.3}$$

Lemma 3.2.1. *Let $t \geq 2$ be the minimal index such that $W_{\eta_t} \nsubseteq \mathbf{1}^{\perp}$. Then there are constants $A_1, A_2 > 0$, depending only on the parameters of the tiling, with the following properties:*

(i) There exists a $j \in \{1, \dots, n\}$ such that for every m and $T \in \mathscr{T}^{(m)}$ of type j

$$A_1 \cdot m^{k_t - 1} |\eta_t|^m \leq |\#(Y \cap T) - \alpha \cdot \lambda(T)|, \tag{3.4}$$

(ii) For every $T \in \mathscr{T}^{(m)}$

$$|\#(Y \cap T) - \alpha \cdot \lambda(T)| \leq A_2 \cdot m^{k_t - 1} |\eta_t|^m, \tag{3.5}$$

where k_t is the size of the maximal Jordan block of η_t in A_H.

Proof. Let $T \in \mathscr{T}^{(m)}$ and write $a_T = \sum_{i=1}^{n} c_i v_i$. Note that $disc(v_1) = 0$, and also $\langle u_1, v_i \rangle = 0$ for every $i \neq 1$. So we have

$$disc(a_T) = \langle \mathbf{1}, \sum_{i=2}^{n} c_i v_i \rangle = \langle \mathbf{1}, \sum_{i=t}^{n} c_i v_i \rangle. \tag{3.6}$$

But, if T in τ_m is of type j then $a_T = A_H^m e_j$. Write $e_j = \sum_{i=1}^{n} b_i v_i$, then

$$a_T = A_H^m \left(\sum_{i=1}^{n} b_i v_i \right) = \sum_{i=1}^{n} b_i A_H^m (v_i).$$

So for every i, $c_i = b_i \cdot Const \cdot m^{k_i - 1} \cdot \eta_i^m$. Considering (3.6), this proves (ii). For (i), if v_t has a Jordan block of size k_t, let v_ℓ the last vector in the corresponding Jordan chain. Note that there exists a j with $b_\ell^{(j)} \neq 0$ in the presentations $e_j = \sum_{i=1}^{n} b_i^{(j)} v_i$. Using (3.6) in the same way again, we showed (i). □

Remark 3.2.2. *By (3.6), if t as above does not exist, then the lemma holds with $\eta_t = 0$.*

3.3 Economic Packing for Patches

We denote by ∂A and $int(A)$ the boundary and interior of a set $A \subseteq \mathbb{R}^d$ with respect to the standard topology of $\mathbb{R}^d$, and by $\|\cdot\|_1$ the standard ℓ_1 norm on $\mathbb{R}^d$.

In this section we prove a series of lemmas that help us estimate the terms that appears in Theorem 3.1.5. Our main objective of this section is to prove Proposition 3.3.5 below. This Proposition gives a very good bound for the number of tiles from each generation τ_m that one needs in order to obtain a given patch in a substitution tiling, using unions and proper differences. Laczkovich proved this proposition for the lattice unit cube tiling in [L92], and here we give a proof for the more general case by generalizing his arguments. Proposition 3.3.5 is the key point for the proof of Theorem 3.1.3 in §3.4. To prove this proposition we had to first prove an isoperimetric lemma, Lemma 3.3.3, to replace the lemma that Laczkovich is using, which is simply false in our more general context.

Lemma 3.3.1. *For every d there is a constant C_1 such that for every $U \subseteq \mathbb{R}^d$, a finite union of unit lattice cubes, and every $R > 0$, we have*

$$\lambda\left(\{x \in U : d(x, \partial U) \leq R\}\right) \leq C_1 \cdot R^d \cdot \lambda_{d-1}(\partial U).$$

Proof. This is a direct consequence of Lemmas 2.1 and 2.2 of [L92]. □

Lemma 3.3.2. *There is a constant C_2, that depends on the parameters of the tiling, such that for any $T \in \mathscr{T}$*

$$\lambda\left(\{x \in \mathbb{R}^d : d(x, \partial T) \leq 1\}\right) \leq C_2 \cdot \lambda_{d-1}(\partial T).$$

Proof. Denote by Q_r the d-dimensional cube with edge of length r. Fix a biLipschitz homeomorphism $\psi_i : \mathcal{T}_i \to Q_1$, denote its biLipschitz constant by K_i, and let $K = \max_i\{K_i\}$. Let $T \in \mathscr{T}$ and suppose that $T \in \mathscr{T}^{(m)}$, a tile of type i. Then by rescaling the picture by ξ^m we get a biLipschitz homeomorphism $\phi : T \to Q_{\xi^m}$, with the same biLipschitz constant. Since ϕ is biLipschitz, it follows that

$$\phi\left(\{x \in T : d(x, \partial T) \leq 1\}\right) \subseteq \{x \in Q_{\xi^m} : d(x, \partial Q_{\xi^m}) \leq K\}.$$

Then

$$\lambda\left(\{x \in T : d(x, \partial T) \leq 1\}\right) \leq K^d \cdot \lambda\left(\{x \in Q_{\xi^m} : d(x, \partial Q_{\xi^m}) \leq K\}\right)$$

Applying the same argument to the tiles which are adjacent to T we obtain

$$\lambda\left(\{x \in \mathbb{R}^d : d(x, \partial T) \leq 1\}\right) \leq K^d \cdot \lambda\left(\{x \in \mathbb{R}^d : d(x, \partial Q_{\xi^m}) \leq K\}\right). \tag{3.7}$$

It also follows that

$$\lambda_{d-1}(\partial Q_{\xi^m}) \leq K^{d-1} \cdot \lambda_{d-1}(\partial T) \tag{3.8}$$

(see Theorem 1 in [EG92] p. 75). Then by (3.7), (3.8), and Lemma 3.3.1 we have

$$\lambda\left(\{x\in\mathbb{R}^d : d(x,\partial T)\le 1\}\right) \overset{(3.7)}{\le} K^d\cdot\lambda\left(\{x\in\mathbb{R}^d : d(x,\partial Q_{\xi^m})\le K\}\right)\le$$

$$C_1\cdot K^{2d}\cdot\lambda_{d-1}(\partial Q_{\xi^m}) \overset{(3.8)}{\le} C_1\cdot K^{3d-1}\cdot\lambda_{d-1}(\partial T).$$

□

For the next lemma, we use the same notations $\mathscr{T}^{(m)}$, $\mathscr{T}$, and $\mathscr{V}$ as defined at the end of §3.2.

Lemma 3.3.3. *Let $T\in\mathscr{T}$, and $c\in(0,\frac{1}{2})$. Then there is an $\varepsilon>0$ such that for any $V\in\mathscr{V}, V\subseteq T$, with $c\cdot\lambda(T)\le\lambda(V)\le\frac{1}{2}\lambda(T)$, we have*

$$\lambda_{d-1}(\partial V\cap int(T))\ge\varepsilon\cdot\lambda_{d-1}(\partial T). \tag{3.9}$$

Proof. This Lemma follows from the relative isoperimetric inequality, see [EG92] p. 190. By this inequality, if B is a closed ball, and $E\subseteq B$ is a closed set of finite perimeter (i.e. χ_E has a bounded variation) then we have

$$\min\{\lambda(E),\lambda(B\setminus E)\}^{\frac{d-1}{d}}\le C\cdot\lambda_{d-1}(\partial E\cap int(B)), \tag{3.10}$$

where C depends only on d. Fix a biLipschitz homeomorphism $\psi_i:\mathcal{T}_i\to B(0,1)$, denote its biLipschitz constant by K_i, and let $K=\max_i\{K_i\}$. Suppose that T is a tile of type i, then by rescaling the picture by ξ^m we get a biLipschitz homeomorphism $\phi: T\to B=B(0,\xi^m)$, with the same biLipschitz constant. Since ϕ is biLipschitz, it follows that

$$\frac{1}{K^d}\lambda(V)\le\lambda(\phi(V))\le K^d\lambda(V)\quad\text{and}\quad\frac{1}{K^{d-1}}\lambda_{d-1}(\phi(\partial V\cap int(T)))\le\lambda_{d-1}(\partial V\cap int(T))$$

(see [EG92] p.75). Considering (3.10) with $E=\phi(V)$ we obtain

$$\lambda_{d-1}(\partial V\cap int(T))\ge\frac{\min\{\lambda(\phi(V)),\lambda(\phi(T\setminus V))\}^{\frac{d-1}{d}}}{C\cdot K^{d-1}}\ge\frac{\min\{\lambda(V),\lambda(T\setminus V)\}^{\frac{d-1}{d}}}{C\cdot(K^{d-1})^2}$$

$$\ge\frac{c^{\frac{d-1}{d}}\cdot\lambda(T)^{\frac{d-1}{d}}}{C\cdot(K^{d-1})^2}=\frac{c^{\frac{d-1}{d}}\cdot s_i^{\frac{d-1}{d}}\cdot\xi^{m(d-1)}}{C\cdot(K^{d-1})^2}=\frac{c^{\frac{d-1}{d}}\cdot s_i^{\frac{d-1}{d}}}{C\cdot(K^{d-1})^2\cdot\lambda_{d-1}(\partial\mathcal{T}_i)}\cdot\lambda_{d-1}(\partial T).$$

Setting $s=\min_i\{s_i\}$ and $D_{\max}=\max_i\{\lambda_{d-1}(\partial\mathcal{T}_i)\}$ we get

$$\varepsilon=\frac{c^{\frac{d-1}{d}}\cdot s^{\frac{d-1}{d}}}{C\cdot(K^{d-1})^2\cdot D_{\max}}, \tag{3.11}$$

that satisfies the assertion, and does not depend on the type of the tile T. □

Corollary 3.3.4. *Let $T \in \mathscr{T}$, $c \in (0, \frac{1}{2})$, and ε as in (3.11). Suppose that $V \in \mathscr{V}, V \subseteq T$ with $\lambda(V) \leq (1-c) \cdot \lambda(T)$ and $\lambda_{d-1}(\partial V \cap int(T)) < \varepsilon \cdot \lambda_{d-1}(\partial T)$, then $\lambda(V) < \frac{1}{2}\lambda(T)$.*

Proof. Assume otherwise, then we have $\lambda_{d-1}(\partial(T \smallsetminus V) \cap int(T)) < \varepsilon \cdot \lambda_{d-1}(\partial T)$ and $c \cdot \lambda(T) \leq \lambda(T \smallsetminus V) \leq \frac{1}{2}\lambda(T)$, contradicting Lemma 3.3.3. □

For a T in τ_m we denote by T^* the unique tile of τ_{m+1} that contains T. We denote $\rho = \frac{\max_i\{s_i\}}{\min_i\{s_i\}} \geq 1$, then for any tile $T \in \mathscr{T}$ we have

$$\rho^{-1} \cdot \xi^{-d} \leq \frac{\lambda(T)}{\lambda(T^*)} \leq \rho \cdot \xi^{-d} \tag{3.12}$$

For a set $X \subseteq \mathscr{T}$ we denote by $S(X)$ the closure of X under the operations of disjoint union and proper difference, where every element of X can be used only once. For the following lemma we set ε as in Lemma 3.3.3 and define the following constants:

$$D_{\min} = \min_i \{\lambda_{d-1}(\partial \mathcal{T}_i)\}, \quad C = \frac{\rho \cdot \xi(\rho \cdot \xi^d + 1)}{\varepsilon \cdot D_{\min}} \quad \text{and} \quad c = (2\rho)^{-1} \cdot \xi^{-d} \in \left(0, \frac{1}{2}\right). \tag{3.13}$$

Proposition 3.3.5. *Let*

$$V \in \mathscr{V}, \quad T \in \mathscr{T}, \quad V \subseteq T, \quad \text{and} \quad \lambda(V) \leq \frac{1}{2}\lambda(T). \tag{3.14}$$

Then there exists $T_1, \ldots, T_n \in \mathscr{T}$, $T_i \subseteq T$ for all i, such that $V \in S(\{T_1, \ldots, T_n\})$, and for every m we have:

$$\#\{i : T_i \in \mathscr{T}^{(m)}\} \leq C \cdot \frac{\lambda_{d-1}(\partial V \cap int(T))}{\xi^{m(d-1)}}.$$

Proof. The proof is by induction on m, where $T \in \mathscr{T}^{(m)}$. If $m = 0$ then $\lambda(V) \leq \frac{1}{2}\lambda(T)$ implies that $V = \varnothing$, so the assertion is obvious. Assume the assertion for any $T \in \mathscr{T}^{(m)}$ with $m < m_0$, and let V and T be as in (3.14) with $T \in \mathscr{T}^{(m_0)}$. Consider the following collection of tiles:

$$\mathscr{A} = \left\{ P \in \mathscr{T} : \begin{array}{c} P \subseteq T, \\ \lambda(P \cap V) \geq c \cdot \lambda(P), \\ \lambda_{d-1}(\partial V \cap int(P)) < \varepsilon \cdot \lambda_{d-1}(\partial P) \end{array} \right\},$$

where ε is as in Lemma 3.3.3 and c is as in (3.13) (it might be that $\mathscr{A} = \varnothing$). Note that every $P \in \mathscr{A}$ satisfies:

$$\lambda(P \smallsetminus V) \leq (1-c)\lambda(P), \quad \text{and} \quad \lambda_{d-1}(\partial(P \smallsetminus V) \cap int(P)) < \varepsilon \cdot \lambda_{d-1}(\partial P).$$

Then by Corollary 3.3.4 we have

$$\lambda(P \setminus V) < \frac{1}{2}\lambda(P). \tag{3.15}$$

Let $P_1, \dots, P_\ell$ be the maximal elements of $\mathscr{A}$ (w.r.t. inclusion). Then $\bigcup \mathscr{A} = \bigcup_{j=1}^{\ell} P_j \subseteq T$, and $P_1, \dots, P_\ell$ has pairwise disjoint interiors. Denote $V_1 = V \cup \bigcup_{j=1}^{\ell} P_j$. Then

$$\lambda(V_1) \leq \lambda(V) + \sum_{j=1}^{\ell} \lambda(P_j \setminus V) \overset{(3.15)}{<} \frac{1}{2}\lambda(T) + \sum_{j=1}^{\ell} \frac{1}{2}\lambda(P_j) \leq \lambda(T). \tag{3.16}$$

Note that if $\mathscr{A} = \varnothing$ we only get $\leq$ in the middle inequality, but then the last inequality is strict. Thus $V_1 \subsetneqq T$, and in particular $P_j \subsetneqq T$ for every j. By (3.15), we may apply the induction hypothesis for $P_j \setminus V \in \mathscr{V}$ and the tile P_j, to obtain tiles $T_{j1}, \dots, T_{jn_j}$ such that $T_{jr} \subseteq P_j$, $P_j \setminus V \in S(\{T_{j1}, \dots, T_{jn_j}\})$, and for every m we have:

$$\# \left\{ r : T_{jr} \in \mathscr{T}^{(m)} \right\} \leq C \cdot \frac{\lambda_{d-1}(\partial V \cap int(P_j))}{\xi^{m(d-1)}}. \tag{3.17}$$

Now let $T_1, \dots, T_n$ be the maximal tiles that are contained in V_1. Then $T_1, \dots, T_n$ has pairwise disjoint interiors and their union is equal to V_1. So we can write

$$V = V_1 \setminus \bigcup_{j=1}^{\ell} (P_j \setminus V) = \left(\bigcup_{i=1}^{n} T_i \right) \setminus \bigcup_{j=1}^{\ell} (P_j \setminus V),$$

where the sets $P_j \setminus V$ are pairwise disjoint. This implies that

$$V \in S(\{T_1, \dots, T_n, T_{11}, \dots, T_{1n_1}, \dots, T_{\ell 1}, \dots, T_{\ell n_\ell}\}).$$

Fix $m \in \mathbb{N}$ and denote $E = \{i : T_i \in \mathscr{T}^{(m)}\}$, and $E_j = \{r : T_{jr} \in \mathscr{T}^{(m)}\}$. It remains to show that

$$|E| + \sum_{j=1}^{\ell} |E_j| \leq C \cdot \frac{\lambda_{d-1}(\partial V \cap int(T))}{\xi^{m(d-1)}}. \tag{3.18}$$

We first estimate $|E|$. Fix an $i \in E$. Since T_i is maximal in V_1, if follows that $T_i^* \nsubseteq V_1$. In particular, by the definition of V_1, since the P_j's are maximal in $\mathscr{A}$, we have $T_i^* \notin \mathscr{A}$. by (3.16), $V_1 \subsetneqq T$, then $T_i \subsetneqq T$, and therefore $T_i^* \subseteq T$. Our next goal is to show that

$$\lambda(T_i^* \cap V) \geq c \cdot \lambda(T_i^*). \tag{3.19}$$

If $int(T_i) \cap \left(\bigcup_{j=1}^{\ell} int(P_j) \right) = \varnothing$ then $T_i \subseteq V$, and therefore

$$\lambda(T_i^* \cap V) \geq \lambda(T_i) \overset{(3.12)}{\geq} \rho^{-1} \cdot \xi^{-d} \cdot \lambda(T_i^*) \overset{(3.13)}{>} c \cdot \lambda(T_i^*).$$

Otherwise, $int(T_i)$ intersect $int(P_j)$ for some j. Then either $T_i \subsetneqq P_j$ or $P_j \subseteq T_i$. If $T_i \subsetneqq P_j$ then $T_i^* \subseteq P_j \subseteq V_1$, a contradiction. Then $P_j \subseteq T_i$ whenever $int(T_i) \cap int(P_j) \neq \varnothing$. Denote by J the set of indices j such that $P_j \subseteq T_i$, then we have

$$\lambda(T_i \smallsetminus V) \leq \lambda\left(\bigcup_{j\in J}(P_j \smallsetminus V)\right) \leq \sum_{j\in J}\lambda(P_j \smallsetminus V)$$

$$\overset{(3.15)}{<} \sum_{j\in J}\frac{1}{2}\lambda(P_j) \leq \frac{1}{2}\lambda(T_i).$$

Hence

$$\lambda(T_i^* \cap V) \geq \lambda(T_i \cap V) > \frac{1}{2}\lambda(T_i) \overset{(3.12)}{\geq} (2\rho)^{-1} \cdot \xi^{-d} \cdot \lambda(T_i^*) \overset{(3.13)}{=} c \cdot \lambda(T_i^*).$$

Thus (3.19) holds. Since $T_i^* \subseteq T$ and $T_i^* \notin \mathscr{A}$, it follows form (3.19) and from the definition of $\mathscr{A}$ that $T_i^* \notin \mathscr{A}$ because it satisfies

$$\lambda_{d-1}(\partial V \cap int(T_i^*)) \geq \varepsilon \cdot \lambda_{d-1}(\partial T_i^*). \tag{3.20}$$

Let $K = \partial V \cap \bigcup_{i\in E} int(T_i^*)$. Since the T_i's are distinct elements of $\mathscr{T}^{(m)}$, and by (3.12), each point of K is covered by at most $\rho \cdot \xi^d \quad T_i^*$'s. Therefore, by (3.20), we have

$$\rho\cdot\xi^d\lambda_{d-1}(K) \geq \sum_{i\in E}\lambda_{d-1}(K\cap T_i^*) = \sum_{i\in E}\lambda_{d-1}(\partial V\cap int T_i^*) \overset{(3.20)}{\geq} \varepsilon\cdot\lambda_{d-1}(\partial T_i^*)\cdot|E|,$$

and hence

$$|E| \leq \frac{\rho\cdot\xi^d}{\varepsilon\cdot\lambda_{d-1}(\partial T_i^*)}\lambda_{d-1}(K). \tag{3.21}$$

Now define

$$J_1 = \{j : P_j \subseteq T_i^* \text{ for some } i \in E\}, \quad \text{and} \quad J_2 = \{1, \ldots, \ell\} \smallsetminus J_1.$$

If $j \in J_1$ and $r \in E_j$ then $T_{jr} \subseteq T_i^*$ for some i. Since T_i^* contains at most $\rho \cdot \xi^d$ tiles of $\mathscr{T}^{(m)}$ we have

$$\sum_{j\in J_1}|E_j| \leq \rho\cdot\xi^d\,|E|.$$

If $j \in J_2$ and $i \in E$ then $int(P_j) \cap int(T_i^*) = \varnothing$ (since $T_i^* \nsubseteq P_j$). Then the set $K_j = \partial V \cap int(P_j)$ is disjoint from K. By (3.17) we have $|E_j| \leq C \cdot \frac{\lambda_{d-1}(K_j)}{\xi^{m(d-1)}}$, and hence

$$\sum_{j=1}^{\ell}|E_j| = \sum_{j\in J_1}|E_j| + \sum_{j\in J_2}|E_j| \leq \rho\cdot\xi^d\,|E| + C\cdot\frac{\lambda_{d-1}\left(\bigcup_{j\in J_2}K_j\right)}{\xi^{m(d-1)}}. \tag{3.22}$$

The sets K and $\bigcup_{j\in J_2} K_j$ are disjoint, and their union is a subset of $\partial V\cap int(T)$, hence

$$|E|+\sum_{j=1}^{\ell}|E_j| \overset{(3.22)}{\leq} (\rho\cdot\xi^d+1)\,|E|+\sum_{j\in J_2}|E_j| \overset{(3.21),(3.22)}{\leq} \frac{\rho\cdot\xi^d(\rho\cdot\xi^d+1)}{\varepsilon\cdot\lambda_{d-1}(\partial T_i^*)}\lambda_{d-1}(K)+C\cdot\frac{\lambda_{d-1}\left(\bigcup_{j\in J_2}K_j\right)}{\xi^{m(d-1)}}$$

$$\overset{(3.13)}{\leq} \frac{\rho\cdot\xi^d(\rho\cdot\xi^d+1)}{\varepsilon\cdot D_{\min}\cdot\xi^{(m+1)(d-1)}}\lambda_{d-1}(K)+C\cdot\frac{\lambda_{d-1}\left(\bigcup_{j\in J_2}K_j\right)}{\xi^{m(d-1)}}$$

$$\overset{(3.13)}{\leq} \frac{C}{\xi^{m(d-1)}}\left(\lambda_{d-1}(K)+\lambda_{d-1}\left(\bigcup_{j\in J_2}K_j\right)\right)\leq C\cdot\frac{\lambda_{d-1}(\partial V\cap int(T))}{\xi^{m(d-1)}}.$$

Thus (3.18) holds and the proof is complete. □

3.4 Proof of the Main Theorems

Proof of Theorem 3.1.3. Proof of (I): We show that if $|\eta_t|>\eta_1^{\frac{d-1}{d}}$ then (i) of Theorem 3.1.5 does not hold for any α. Fix a $j\in\{1,\dots,n\}$ such that (3.4) holds for any tile of type j, and consider the sequence of measurable sets $T^{(m)}$, the m'th inflation of $\mathcal{T}_j$. Then by Lemma 3.3.2 we have

$$\lambda\left(\{x\in\mathbb{R}^d : d(x,\partial T)\leq 1\}\right)\leq C_2\cdot\lambda_{d-1}(\partial T)=C_2\cdot\lambda_{d-1}(\partial\mathcal{T}_j)\cdot\xi^{m(d-1)}.$$

Recall that $\xi^d=\eta_1$, then we have

$$\lambda\left(\{x\in\mathbb{R}^d : d(x,\partial T)\leq 1\}\right)\leq C_2\left(\eta_1^{\frac{d-1}{d}}\right)^m\lambda_{d-1}(\partial\mathcal{T}_i). \tag{3.23}$$

As we did in the proof of Lemma 3.2.1, for any α different than the one defined in (3.1) $disc(v_1)\neq 0$, and so $disc(a_T)=Const\cdot\eta_1^m$. For large m's, this is obviously greater than any constant times $\lambda\left(\{x\in\mathbb{R}^d : d(x,\partial T)\leq 1\}\right)$. For α as in (3.1), by Lemma 2.2.1 we have

$$\left|\#(Y\cap T^{(m)})-\alpha\cdot\lambda(T^{(m)})\right|\geq A_1\cdot|\eta_t|^m,$$

which by assumption is greater than $Const\left(\eta_1^{\frac{(d-1)}{d}}\right)^m$, for any constant and for a large enough m's. Considering (3.23), we proved that (i) of Theorem 3.1.5 does not hold.

Proof of (II): We show that (ii) of Theorem 3.1.5 holds, where α is as in (3.1). Let $R=\left\lceil\max_i\{diam(\mathcal{T}_i)\}\right\rceil$, where $diam(A)$ denote the diameter of a set A. It is sufficient to show that (ii) holds for any U, a finite union of R-cubes (cubes with edge length R and corners at $R\cdot\mathbb{Z}^d$). Let U be a finite union of

R-cubes. For every $y \in Y$ we denote by T_y the tile of τ_0 that corresponds to y, and define an $V \in \mathscr{V}$ by $V = \bigcup\{T_y : y \in U\}$. Then $U \subseteq V \cup (U \smallsetminus V)$. Note that $U \smallsetminus V \subseteq \{x \in U : d(x, \partial U) \leq R\}$, so it follows from Lemma 3.3.1 that

$$\lambda(U \smallsetminus V) \leq C_1 \cdot R^d \cdot \lambda_{d-1}(\partial U).$$

Since $\#(U \cap Y) = \#(V \cap Y)$ we have

$$|\#(U \cap Y) - \alpha \cdot \lambda(U)| \leq |\#(V \cap Y) - \alpha \cdot \lambda(V)| + \alpha \cdot C_1 \cdot R^d \cdot \lambda_{d-1}(\partial U). \tag{3.24}$$

So it is enough to estimate $|\#(V \cap Y) - \alpha \cdot \lambda(V)|$.

Next we claim that $\partial V \subseteq \{x \in \mathbb{R}^d : d(x, \partial U) \leq R\}$. Indeed, if $x \in \partial V$ then either $x \in U$ or $x \notin U$. If $x \in U$, since $x \in \partial V$, $x \in \partial T_y$ for some $y \notin U$, and therefore $d(x, \partial U) \leq d(x, y) \leq diam(T_y) \leq R$. A similar argument holds if $x \notin U$ since x also belong to ∂T_y for some $y \in U$. Therefore, every tile T of τ_0 with $T \cap \partial V \neq \varnothing$ is contained in $\{x \in \mathbb{R}^d : d(x, \partial U) \leq 2R\}$. Denote by $C_3 = \max_i \frac{\lambda_{d-1}(\partial \mathcal{T}_i)}{\lambda(\mathcal{T}_i)}$. Then by Lemma 3.3.1 we have

$$\begin{aligned} &\lambda_{d-1}(\partial V) \leq \sum_{\substack{T \in \mathscr{T}^{(0)} \\ T \cap \partial V \neq \varnothing}} \lambda_{d-1}(\partial T) \leq \sum_{\substack{T \in \mathscr{T}^{(0)} \\ T \cap \partial V \neq \varnothing}} C_3 \cdot \lambda(T) \leq \\ &C_3 \cdot \lambda\left(\{x \in \mathbb{R}^d : d(x, \partial U) \leq 2R\}\right) \leq C_3 \cdot C_1 \cdot (2R)^d \cdot \lambda_{d-1}(\partial U). \end{aligned} \tag{3.25}$$

To finish the proof, we apply Proposition 3.3.5 to V. We pick a large enough $T \in \mathscr{T}$ such that (3.14) holds. By Proposition 3.3.5 we obtain $T_1, \dots, T_n \in \mathscr{T}$ such that $V \in S(\{T_1, \dots, T_n\})$, and for every m we have:

$$\#\{i : T_i \in \mathscr{T}^{(m)}\} \leq C \cdot \frac{\lambda_{d-1}(\partial V \cap int(T))}{\xi^{m(d-1)}}. \tag{3.26}$$

Note that if $A, B \in \mathscr{V}$ and $int(A) \cap int(B) = \varnothing$ then

$$\#(Y \cap (A \cup B)) - \alpha \cdot \lambda(A \cup B) = \#(Y \cap A) - \alpha \cdot \lambda(A) + \#(Y \cap B) - \alpha \cdot \lambda(B),$$

and similarly if $B \subseteq A$ then

$$\#(Y \cap (A \smallsetminus B)) - \alpha \cdot \lambda(A \smallsetminus B) = \#(Y \cap A) - \alpha \cdot \lambda(A) - (\#(Y \cap B) - \alpha \cdot \lambda(B)).$$

Therefore, since $V \in S(\{T_1, \dots, T_n\})$, we have

$$|\#(Y \cap V) - \alpha \cdot \lambda(V)| \leq \sum_{i=1}^{n} |\#(Y \cap T_i) - \alpha \cdot \lambda(T_i)| \leq \sum_{m=0}^{\infty} \sum_{T_i \in \mathscr{T}^{(m)}} |\#(Y \cap T_i) - \alpha \cdot \lambda(T_i)|$$

$$\overset{(3.5),(3.26)}{\leq} \sum_{m=0}^{\infty} \left[C \cdot \frac{\lambda_{d-1}(\partial V \cap int(T))}{\xi^{m(d-1)}} \cdot A_2 \cdot m^{k_t - 1} |\eta_t|^m\right] \leq \left[\sum_{m=0}^{\infty} \frac{m^{k_t-1} |\eta_t|^m}{(\xi^{d-1})^m}\right] \cdot C \cdot A_2 \cdot \lambda_{d-1}(\partial V).$$

By the assumption, $|\eta_2| < \eta_1^{\frac{d-1}{d}} = \xi^{d-1}$, and therefore the series converges and we have

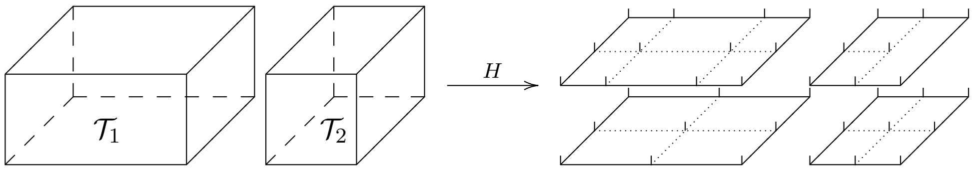
$$|\#(Y \cap V) - \alpha \cdot \lambda(V)| \leq Const \cdot \lambda_{d-1}(\partial V).$$

Considering (3.24) and (3.25), we have shown (ii) of Theorem 3.1.5, which implies the assertion.

Proof of (III): Under the assumption of $k_t > 1$, using (3.4) in the same way as in the proof of (I), yields the assertion.

For the case where the Jordan block of η_t is trivial, we give an example in $\mathbb{R}^3$, where the corresponding separated net is not a BDD of $\mathbb{Z}^3$.

Example : Consider the substitution rule H that is defined by this picture:

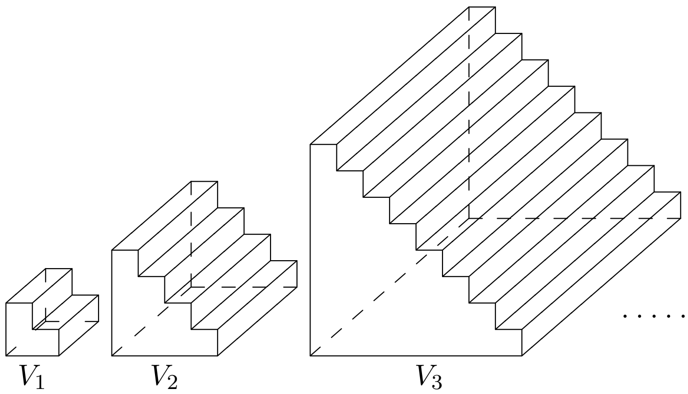


So we have $A_H = \begin{pmatrix} 6 & 1 \\ 4 & 6 \end{pmatrix}, d = 3, \eta_1 = 8$, and $\eta_2 = 4 = \eta_1^{(d-1)/d}$. Denote by $T_i^{(m)}, i = 1, 2$ a tile of type i in $\mathscr{T}^{(m)}$. For every $m \in \mathbb{N}$ we define a patch $V_m \in \mathscr{V}$ in the following process:

- Take a tile T_2^{m+1} and remove from it the (unique) $T_1^{(m)}$ that it contains.
- From what is left $U_1^{(1)}$, remove all the $T_1^{(m-1)}$ with at least two faces common with $\partial U_1^{(1)}$.

⋮

- Eventually, from $U_1^{(m-1)}$ remove all the $T_1^{(1)}$ with at least two faces common with $\partial U_1^{(m-1)}$, to get $U_1^{(m)}$. Define $V_m = U_1^{(m)}$.



So obviously

$$\lambda_2(\partial V_m) \leq \lambda_2(\partial T_2^{(m+1)}) = 6 \cdot 4^m. \tag{3.27}$$

We fix an m and estimate $|\#(Y \cap V_m) - \alpha \cdot \lambda(V_m)|$. For that we consider the following partition of V_m to tiles from different levels $\mathscr{T}^{(k)}$'s:

$$
\begin{aligned}
&\mathscr{U}_m = \{T \in \mathscr{T}^{(m)} : int(T) \subseteq V_m\} \\
&\mathscr{U}_{m-1} = \{T \in \mathscr{T}^{(m-1)} : int(T) \subseteq V_m \setminus \bigcup \mathscr{U}_m\} \\
&\vdots \\
0 \le k < m : \quad &\mathscr{U}_k = \{T \in \mathscr{T}^{(k)} : int(T) \subseteq V_m \setminus \bigcup (\mathscr{U}_{k+1} \cup \ldots \cup \mathscr{U}_m)\}
\end{aligned}
$$

For $i = 1, 2$ and $k \in \{0, 1, \ldots, m\}$ let $t_{i,k} = \#\{T_i^{(k)} \in \mathscr{U}_k\}$. By the construction,

$$
t_{1,k} = 0 \text{ for all } k, \text{ and } t_{2,k} = \begin{cases} 2 \cdot 4^{m-k-1}, & k \neq 0 \\ 6 \cdot 4^{m-1} & k = 0 \end{cases}. \tag{3.28}
$$

Recall that the discrepancy of V_m depends only on the vector $a_{V_m} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ (see (3.2)). We can write it now in terms of the $t_{2,k}$'s. Calculations of $A_H^k e_2$ shows that:

$$
\begin{aligned}
a_1 &= \sum_{k=0}^{m} t_{2,k} \cdot A_H^k e_2(1) = \sum_{k=0}^{m} \frac{1}{4} \cdot t_{2,k}(8^k - 4^k), \\
a_2 &= \sum_{k=0}^{m} t_{2,k} \cdot A_H^k e_2(2) = \sum_{k=0}^{m} \frac{1}{2} \cdot t_{2,k}(8^k + 4^k)
\end{aligned} \tag{3.29}
$$

Note that $\alpha = 3/4$ (see (3.1)), then

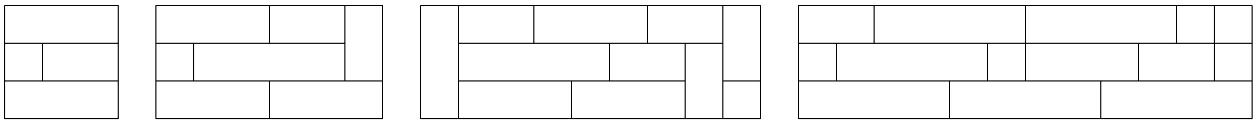
$$
\begin{aligned}
&|\#(Y \cap V_m) - \alpha \cdot \lambda(V_m)| = \left| a_1 + a_2 - \frac{3}{4}(2a_1 + a_2) \right| = \left| \frac{1}{4} a_2 - \frac{1}{2} a_1 \right| \\
&\overset{(3.29)}{=} \frac{1}{4} \left| \sum_{k=0}^{m} t_{2,k} \cdot 4^k \right| \overset{(3.28)}{=} \frac{1}{4} \left| 6 \cdot 4^{m-1} + \sum_{k=1}^{m} 2 \cdot 4^{m-k-1} \cdot 4^k \right| = \left(\frac{m+3}{8} \right) 4^m.
\end{aligned} \tag{3.30}
$$

Observe that (3.27) and (3.30) together shows that (ii) of Theorem 3.1.5 does not hold, which implies that any tiling in X_H correspond to a separated net which is not a BDD of $\mathbb{Z}^3$. □

3.5 Examples

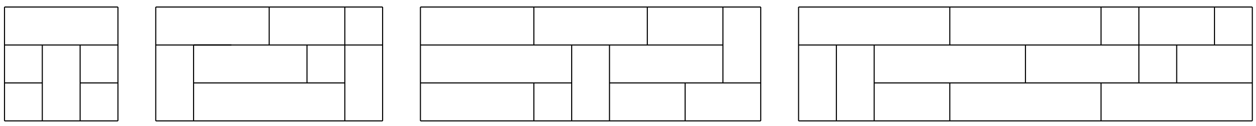
In this last section we give some examples for primitive substitution tilings to show that the different cases that appears in Theorem 3.1.3 exists. In all of the examples below we give the substitution H and refer the result to any separated net that corresponds to any substitution tiling in X_H. Note that in all the examples below the order of the tiles does not matter, but only how many we have of each type. We add the drawings of the substitution rule in order to show that there are substitutions that correspond to the matrices.

Example 3.5.1.



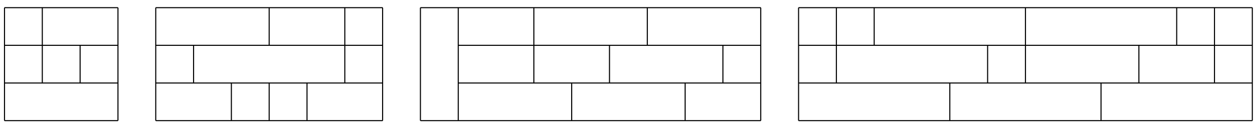
$A_H = \begin{pmatrix} 1 & 1 & 1 & 5 \\ 1 & 2 & 5 & 2 \\ 2 & 3 & 4 & 1 \\ 0 & 1 & 1 & 6 \end{pmatrix}$. *The eigenvalues are:* $9, 4, 1, -1$, *and we have* $4 > 9^{1/2}$. *But the eigenvector that corresponds to* 4 *is in* $\boldsymbol{1}^{\perp}$, *then* $\eta_t = 1 < 9^{1/2}$, *and therefore any tiling in* X_H *give rise to a separated net which is BDD to* $\mathbb{Z}^2$.

Example 3.5.2.



$A_H = \begin{pmatrix} 4 & 3 & 1 & 3 \\ 1 & 4 & 5 & 5 \\ 1 & 1 & 4 & 1 \\ 0 & 1 & 1 & 5 \end{pmatrix}$. *The eigenvalues are:* $9, 3, 3, 2$, *where* 3 *has a non-trivial Jordan block, and the generalized eigenvector is not in* $\boldsymbol{1}^{\perp}$. *Here, by Remark 3.1.4, any separated net that correspond to a tiling in* X_H *is not BDD to* $\mathbb{Z}^2$.

Example 3.5.3.



$A_H = \begin{pmatrix} 4 & 5 & 1 & 7 \\ 1 & 3 & 4 & 1 \\ 1 & 1 & 6 & 1 \\ 0 & 1 & 0 & 6 \end{pmatrix}$. *The eigenvalues are:* $9, 5, 3, 2$, *and we have* $5 > 9^{1/2}$. *But the eigenvector that corresponds to* 5 *is in* $\boldsymbol{1}^{\perp}$, *then* $\eta_t = 3 = 9^{1/2}$. *Then we have here an example for a substitution that we don't know to say whether the corresponding separated nets are BDD to* $\mathbb{Z}^2$ *or not.*

Chapter 4

The Danzer Problem

4.1 Introduction

We use the following standard notations for lower and upper bounds of functions:

$$f(x) = O(g(x)) \quad \text{means that} \quad \exists K_1, x_0 \ \forall x \geq x_0 : \ f(x) \leq K_1 g(x)$$
$$f(x) = \Omega(g(x)) \quad \text{means that} \quad \exists K_2, x_0 \ \forall x \geq x_0 : \ f(x) \geq K_2 g(x)$$

This chapter deals with the following question that was originally asked by L. Danzer in the sixties, see [D65].

Question 4.1.1. *Is there a set $D \subseteq \mathbb{R}^d, (d > 1)$ with growth rate $O(T^d)$, that intersect every convex set of volume 1?*

We say that a discrete set $S \subseteq \mathbb{R}^d$ is of *growth rate* $O(g(T))$ if $\#(S \cap B_T) = O(g(T))$, where B_T is the ball of radius T around the origin. We may also replace balls of radius T by squares of edge length T.

For $d = 1$ Question 4.1.1 is trivial, since a convex set of volume 1 is simply an interval of length 1. For any $d \geq 2$ the question is open, and as we show in §4.2 it is equivalent to a difficult combinatorial question.

A set $D \subseteq \mathbb{R}^d$ that intersect every convex set of volume 1 is called a *Danzer set*. The constant 1 is just a convenient normalization. One could also define a *C-Danzer set*, as a set that intersect every convex set of volume C, and asking whether a C-Danzer set of growth rate $O(T^d)$ exists. But these two questions are obviously equivalent.

Note that the growth rate of a discrete Danzer set is at least $\Omega(T^d)$, since the cube of edge length T contains T^d disjoint unit cubes. Also, as was pointed out by Gowers in [Go00], and as follows from Claim 4.2.4, in Question 4.1.1 it is enough to consider boxes instead of convex sets. By a *box* in $\mathbb{R}^d$ we simply mean the image of a set of the form $[a_1, b_1] \times \ldots \times [a_d, b_d]$ under an isometry $g \in O(d) \ltimes \mathbb{R}^d$. In view of the above, we say that $D \subseteq \mathbb{R}^d$ is a *Danzer set* if it intersect every box of volume 1.

The Danzer problem appears in the literature in several places, and slightly different versions of it were asked by several people. Boshernitzan posed the question whether there is a Danzer set which is also a separated net. In [Go00], Gowers asked if there is a set $D \subseteq \mathbb{R}^d$, with a constant C, such that for every box R of volume 1 we have $1 \leq \#(D \cap R) \leq C$.

Although the Danzer question can be phrased in just one line, and no prior knowledge is required to understand it, there are very few previous results and references regarding it. There are two short discussions on it in [GL87] and [Go00], and the two main previous results appear in [BW71]. We describe these results in the next two theorems.

Theorem 4.1.2 ([BW71], Theorem 1)**.** *A union of grids in $\mathbb{R}^d$ (i.e. translated lattices) is not a Danzer set.*

Theorem 4.1.3 ([BW71], Theorem 2)**.** *There exists a Danzer set $D \subseteq \mathbb{R}^d$ of growth rate $O(T^d(\log T)^{d-1})$.*

Our main results of this chapter are Theorems 4.1.4 and 4.1.5 below, and they are of similar flavor like the results in [BW71]. In Theorem 4.1.4 we have negative results, parallel to Theorem 4.1.2, and Theorem 4.1.5 gives a positive result that improves Theorem 4.1.3.

Theorem 4.1.4 is about certain separated nets arises from substitution tilings (see §1.4 to recall the definitions). Note that if τ is a substitution tiling with prototiles $\mathcal{F} = \{T_1, \ldots, T_n\}$, a function h that picks a point in each prototile defines a separated net that has one point in every tile of τ. Denote this separated net by $Z_{\tau,h}$.

Theorem 4.1.4. *Let τ be a primitive substitution tiling of $\mathbb{R}^d$ with finitely many polygonal prototiles $\mathcal{F} = \{T_1, \ldots, T_n\}$.*

(i) For any function $h : \mathcal{F} \to \bigcup \mathcal{F}$, with $h(T_i) \in T_i$ for every i, $Z_{\tau,h}$ is not a Danzer set.

(ii) Let $Y \subseteq \mathbb{R}^d$ be a random set that is obtained by choosing one point in each tile of τ randomly and independently, with respect to some distribution on each prototile. Then almost surely Y is not a Danzer set.

Theorem 4.1.5. *There exists a Danzer set $D \subseteq \mathbb{R}^d$ of growth rate $O(T^d \log T)$.*

Theorem 4.1.5 is in fact a straight forward corollary of Proposition 4.3.1, together with previous results from combinatorics. This proposition has its own interest since it shows that Question 4.1.1 is equivalent to another question that was well studied in combinatorics and computational geometry. We give the relevant background and state the equivalent question in §4.2, and then prove Proposition 4.3.1, that shows the equivalence, in §4.3. In §4.4 we present the previous combinatorial results and deduce Theorem 4.1.5. After that we prove Theorem 4.1.4 in §4.5.

4.2 An Equivalent Combinatorial Question

In order to describe the combinatorial question that we claim to be equivalent to Question 4.1.1, we first present a couple of notions, taken from computational geometry.

Definition 4.2.1. A *range space* is a pair $(X, \mathcal{S})$ where X is a set, and $\mathcal{S} \subseteq \mathcal{P}(X)$, a set of subsets of X. The elements of X are called *points*, and the elements of $\mathcal{S}$ are called *ranges*.

We remark that this notion is also called a *set system*, or a *hypergraph*, where X is the set of vertices, and S is the set of hyperedges.

Many of the examples that people study are geometric. For example, X is $\mathbb{R}^d$ or $[0,1]^d$, and $\mathcal{S}$ is the set of geometric figures, like half spaces, triangles, align boxes, convex sets, etc. The same ranges are often considered for finite sets $X \subseteq \mathbb{R}^d$, and then saying that $\mathcal{S}$ is the set of half spaces, for example, means that $\mathcal{S} = \{H \cap X : H \subseteq \mathbb{R}^d \text{ is a half space}\}$.

Definition 4.2.2. Let $(X, \mathcal{S})$ be a range space with $\#X = n$. For a given $\varepsilon > 0$, a set $N_\varepsilon \subseteq X$ is called an *ε-net* if for every $S \in \mathcal{S}$ with $\#(S \cap X) \geq \varepsilon n$ we have $S \cap N \neq \varnothing$. The same definition can be made for an infinite set X, replacing n by $\mu(X)$, where μ is a given probability measure on X.

Notice that the notion of an ε-net resemble the notion of a Danzer set, when we take $X = [0,1]^d$ with the standard Lebesgue measure, and S to set of convex subsets of X with volume ε. So a parallel question to Question 4.1.1 in this context is the following:

Question 4.2.3. *For $\varepsilon > 0$ let $X = [0,1]^d$, $\mathcal{S}_\varepsilon = \{\text{convex sets of volume } \varepsilon\}$. Is there an ε-net of size $O(1/\varepsilon)$ for the pair $(X, \mathcal{S}_\varepsilon)$, for every $\varepsilon > 0$?*

As pointed out by Gowers in [Go00], and as the following claim shows, it would be equivalent to ask this question with $\mathcal{S}_\varepsilon = \{\text{boxes of volume } \varepsilon\}$ (not align boxes).

Claim 4.2.4. *There is a constant α_d, that depend only on d, such that for any convex set $K \subseteq \mathbb{R}^d$ there are boxes $R_1 \subseteq K \subseteq R_2$ with $Vol(R_2)/Vol(R_1) \leq \alpha_d$.*

Proof. The assertion follows from John's Theorem, see [B97] Lecture 3, saying that any convex set $K \subseteq \mathbb{R}^d$ contains a maximal ellipsoid B, and that $B \subseteq K \subseteq dB$. Taking R_1 to be the box that is inscribed in B, R_2 the box that circumscribe dB, and $\alpha_d = d^{3d}$, finishes the proof. □

As we show in Claim 4.3.2, a simple rescaling argument shows that the estimate in Question 4.2.3 on the size of an ε-net is indeed the parallel question to ask. Our main result in §4.3 shows that Question 4.1.1 and Question 4.2.3 are actually equivalent. The reason why such a result is interesting is because finding upper and lower bounds for the sizes of ε-nets in various range spaces is a topic that was extensively studied in combinatorics and computational geometry. So we can immediately apply it, and using these combinatorial constructions, to deduce a construction of a Danzer set with a growth rate that beats the one in Theorem 4.1.3.

In the combinatorial result that is related to the Danzer problem the range space is $([0, 1]^d, \{\text{boxes of volume } \varepsilon\})$. This result is a special case of a more general result by Vapnik and Chervonenkis, see [VC71], that gives a construction of a relatively small ε-net wherever the complexity of the ranges in $\mathcal{S}$ is not too high. To make this notion precise, we present the following terminology.

Definition 4.2.5. Let $(X, \mathcal{S})$ be a range space. A set $A \subseteq X$ is called *shattered* if

$$|\{A \cap S : S \in \mathcal{S}\}| = 2^{|A|},$$

where $|A|$ is the cardinality of the set A.

Definition 4.2.6. The *Vapnic Chervonenkis dimension*, or shortly the *VC-dimension*, of a range space $(X, \mathcal{S})$ is

$$VCdim(X, \mathcal{S}) = \sup\{|A| : A \subseteq X \text{ is shattered}\}.$$

For a range space $(X, \mathcal{S})$ and a subset $B \subseteq X$ we denote by $\mathcal{S}|_B = \{s \cap B : s \in \mathcal{S}\}$.

Observe that $VCdim(B, \mathcal{S}|_B) \leq VCdim(X, \mathcal{S})$ for any $B \subseteq X$. More generally, if $\mathcal{S}' \subseteq \mathcal{S}$ then $VCdim(X, \mathcal{S}') \leq VCdim(X, \mathcal{S})$.

Example 4.2.7. *We compute the VC-dimension for several examples, where* $X = [0, 1]^d$*:*

- *$\mathcal{S}$ is the set of closed half-spaces. By half-spaces we mean the set of points in one side of a hyperplane. We show that $VCdim(X, \mathcal{S}) = d + 1$. First note that if $\Lambda \subseteq X$, $\#\Lambda = d+1$, and Λ is in general position, then Λ is shattered. On the other hand, by Radon's Theorem (see [Radon21]), every $\Lambda \subseteq X$ of size $d + 2$ can be divided into two sets A, B such that their convex halls intersect. In particular, A cannot be obtained as $\Lambda \cap H$ for any half-space $H \in \mathcal{S}$.*

- *$\mathcal{S}$ is the set of convex sets. Here $VCdim(X, \mathcal{S}) = \infty$. Let C be a circle, or a $d+1$ dimensional sphere in C. So C is an infinite set (even uncountable). C is shattered, since for any subset A of C the convex hall of A, $conv(A)$, is in $\mathcal{S}$, and $C \cap conv(A) = A$.*

- *$d = 2$, and $\mathcal{S}$ is the set of rectangles (not only align rectangles). We show that $VCdim(X, \mathcal{S}) < 10$. Let $\Lambda \subseteq X$, $\#\Lambda = 10$ and consider the two following cases:*

  (i) If there is some subset $\Lambda' \subsetneqq \Lambda$ with $conv(\Lambda') = conv(\Lambda)$, then clearly $\Lambda' \neq \Lambda \cap S$ for any $S \in \mathcal{S}$.

  (ii) Otherwise, $T = conv(\Lambda)$ is a convex decagon. Let $\{x_0, x_1, \ldots, x_9\}$ be a cyclic order of the vertices of T, we claim that $\Lambda' = \{x_0, x_2, x_4, x_6, x_8\}$ cannot be obtained as $\Lambda \cap S$, for any $S \in \mathcal{S}$. To see this, assume that $\Lambda' = \Lambda \cap S$ for some rectangle $S \in \mathcal{S}$. Then for every $i \in \{0, \ldots, 9\}$ there is an edge of S separating x_i and x_{i+1} (the $+$ is taken $(mod\, 10)$). So the edges of S intersect all the ten edges of the decagon T. But since T is convex, every edge of S can intersect at most two edges of T. A contradiction.

As we mentioned above, the VC-dimension of a range space is a way to measure the complexity of the ranges. The following combinatorial lemma gives one precise meaning to this general idea, and shows that if X is finite, and the VC-dimension is finite, then the number of ranges is polynomial in $\#X$. It was proven originally by Sauer, see [Sa72], and independently by Perles and Shelah. We also refer to [AS08], Lemma 14.4.1, for a short proof.

Lemma 4.2.8. *If $(X, \mathcal{S})$ is a range space with VC-dimension d, and $\#X = n$, then $\#\mathcal{S} \leq \sum_{i=0}^{d} \binom{n}{i}$.*

This Lemma also give rise to the following corollary, see [AS08] Corollary 14.4.3.

Corollary 4.2.9. *Let $(X, \mathcal{S})$ be a range space of VC-dimension d, and let $\mathcal{S}_k = \{s_1 \cap \ldots \cap s_k : s_i \in \mathcal{S}\}$. Then $VCdim(X, \mathcal{S}_k) \leq 2dk \log(dk)$.*

Another result that is relevant to us is the following theorem of Haussler and Welzl.

Theorem 4.2.10 (see [HW87])**.** *Let $(X, \mathcal{S})$ be a range space, where X is finite and $VCdim(X, \mathcal{S}) = d$, then for any $\varepsilon > 0$ there is an ε-net N with $\#N \leq O(\frac{d}{\varepsilon} \log(1/\varepsilon))$.*

For completeness, in §4.4 we give a proof of a special case of this theorem, that we learned from Saurabh Ray.

4.3 The Two Questions are Equivalent

The goal of this section is to prove Proposition 4.3.1. This result in particular implies that Question 4.1.1 and Question 4.2.3 are equivalent (namely, an affirmative answer to one implies an affirmative answer to the other).

Proposition 4.3.1. *For fixed $d \geq 2$, and a function $g(x)$ of polynomial growth, the following are equivalent:*

(i) There exists a Danzer set $D \subseteq \mathbb{R}^d$ with growth rate $O(g(T))$.

(ii) For every $\varepsilon > 0$ there exists an ε-net for $(X, \mathcal{S}_\varepsilon)$ of size $O(g(\varepsilon^{-1/d}))$, where $X = [0,1]^d$ and $\mathcal{S}_\varepsilon = \{$boxes of volume $\varepsilon\}$.

We divide the proof into several parts. The easy implication is (i) implies (ii), as the following simple argument shows.

Claim 4.3.2. *(i) implies (ii).*

Proof. Suppose that $D \subseteq \mathbb{R}^d$ intersect every box of volume 1. For a given $\varepsilon > 0$ consider the square Q of edge length $\varepsilon^{-1/d}$, centered at the origin. Then $N = D \cap Q$ intersect any box of volume 1 contained in Q. Now contract Q (and the whole picture) by a factor of $\varepsilon^{1/d}$ in every axes. Then Q becomes a square of volume 1, and N intersects every box of volume ε in it. So if $D = O(g(T))$, then $\#N = O(g(\varepsilon^{-1/d}))$. □

Corollary 4.3.3. *Fix $C > 0$. If for some $\varepsilon > 0$ a set of C/ε points cannot intersect all boxes of volume ε in $[0,1]^d$, then there is no Danzer set of growth rate $O(T^d)$.*

As mentioned in 4.1, the growth rate of a Danzer set is at least T^d since the cube of edge length T contains T^d disjoint unit cubes. Moreover, we claim that the multiplicative constant C must depend on the dimension d. To be precise, we give the following claim for epsilon nets in $[0,1]^d$, which by Claim 4.3.2 implies the above statement.

Claim 4.3.4. *For any constant $C > 0$, and any function $g(x)$, there is a large enough d such that any set of size $Cg(1/\varepsilon)$ does not intersect all convex sets of volume ε in $[0,1]^d$.*

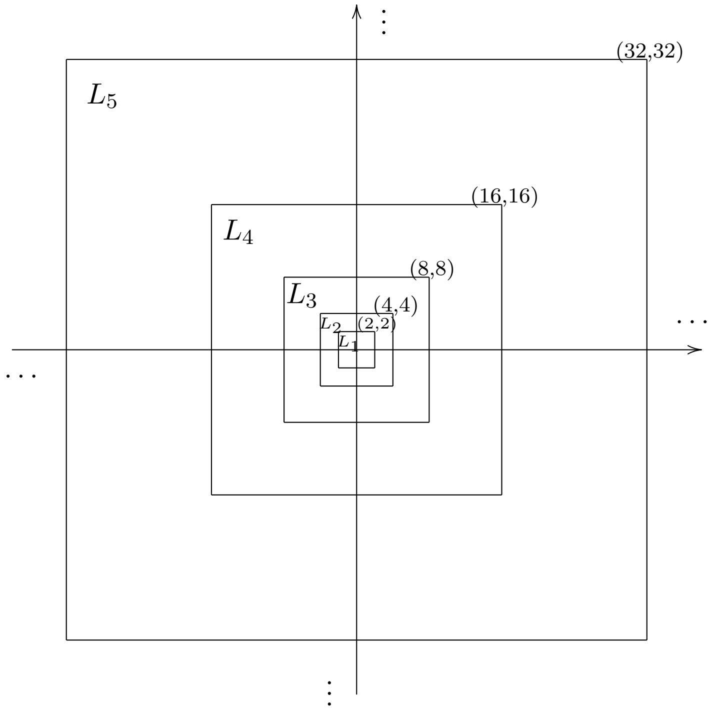
The proof given below is due to Shakhar Smorodinsky.

Proof. For a given $\varepsilon < \frac{1}{2}$, take $d = \lceil Cg(1/\varepsilon) \rceil$. So any $\lceil Cg(1/\varepsilon) \rceil$ points in $[0,1]^d$ are contained in a hyper-plane H. H misses at least half of the cube $[0,1]^d$, so these points clearly miss a convex set of volume at least $\frac{1}{2}$. □

Our next objective is to show that (ii) implies (i). Define the following partition of $\mathbb{R}^d$ into layers:

$$L_1 = [-2,2]^d, \qquad L_i = \left\{ x \in \mathbb{R}^d : 2^{i-1} < \|x\|_\infty \leq 2^i \right\}, \tag{4.1}$$

where $\|\cdot\|_\infty$ is the ℓ_∞ norm.



Set

$$C_d = \frac{1}{4d\log_2(20d)}, \tag{4.2}$$

and denote by

$$Q_t = \{x \in \mathbb{R}^d : \|x\|_\infty \leq t\},$$
$$B_t = \{x \in \mathbb{R}^d : \|x\|_2 \leq t\}.$$

Proposition 4.3.5. *Suppose that for every i we have a set $N_i \subseteq L_i$, that intersects any convex set of volume C_d that is contained in L_i, then*

(i) $D = \bigcup_{i=0}^{\infty} N_i$ *is a Danzer set in* $\mathbb{R}^d$.

(ii) If for every i we have $\#N_i = O(g(2^i))$, then D has growth rate $O(g(T))$.

The proof relies on the following two lemmas.

Lemma 4.3.6. *Let $R \subseteq \mathbb{R}^d$ be a box. Suppose that $Vol(Q_t \cap R) \geq \frac{1}{2} Vol(R)$, then $R \subseteq Q_{5td}$.*

Proof. Let x_0 be a vertex of R, and let $r_1, \ldots, r_d$ be the d edges of R with one end-point at x_0. Denote by $|r|$ the length of a segment r, then we have $Vol(R) = \prod_{i=1}^{d} |r_i|$.

Denote $K = Q_t \cap R$ and define

$$k_i = \text{a segment of maximal length in } K, \text{ in direction } r_i.$$

So clearly $Vol(K) \leq \prod_{i=1}^{d} |k_i|$, and $|k_i| \leq |r_i|$ for all i. Hence by the assumption $\frac{1}{2}Vol(R) \leq Vol(K)$ we have

$$\frac{1}{2}|r_i| \leq |k_i| \tag{4.3}$$

for all i.

Let $\ell = diam(R)$, and let $k \in K$. Since $d(0,k) \leq t\sqrt{d}$, we have

$$R \subseteq B(k,\ell) \subseteq B_{t\sqrt{d}+\ell} \subseteq Q_{t\sqrt{d}+\ell}.$$

On the other hand

$$\ell = diam(R) = \sqrt{|r_1|^2 + \ldots + |r_d|^2} \leq \sqrt{d}\max_i\{|r_i|\} \overset{(4.3)}{\leq} 2\sqrt{d}\max_i\{|k_i|\} \leq 4td.$$

So $R \subseteq Q_{5td}$. □

Lemma 4.3.7. *For any box R of volume 1 in $\mathbb{R}^d$ there is a layer L_i such that $L_i \cap R$ contains a convex set K with $Vol(K) \geq C_d$, where C_d is as in (4.2).*

Proof. Let $m \in \mathbb{N}$ be the minimal integer such that $R \subseteq \bigcup_{i=0}^{m} L_i = Q_{2^m}$. Let $j \in \mathbb{N}$ be the minimal integer satisfying $5d \leq 2^j$. So we may also write

$$Q_{2^m} = Q_{2^{m-j-1}} \cup (L_{m-j} \cup L_{m-j+1} \cup \ldots \cup L_m).$$

Since $Vol(R) = 1$ we either have $Vol(Q_{2^{m-j-1}} \cap R) \geq \frac{1}{2}$ or $Vol((L_{m-j} \cup \ldots \cup L_m) \cap R) \geq \frac{1}{2}$. If $Vol(Q_{2^{m-j-1}} \cap R) \geq \frac{1}{2}$, then by Lemma 4.3.6 we have

$$R \subseteq Q_{2^{m-j-1} \cdot 5d} \subseteq Q_{2^{m-1}} = \bigcup_{i=0}^{m-1} L_i,$$

contradicting the minimality of m. So $Vol((L_{m-j} \cup \ldots \cup L_m) \cap R) \geq \frac{1}{2}$, and therefore $Vol(L_i \cap R) \geq \frac{1}{2(j+1)} \geq \frac{1}{2\log_2(20d)}$ for some $i \in \{m-j, m-j+1, \ldots, m\}$.

Fix i as above, so it is left to find a convex set $K \subseteq L_i \cap R$ with $Vol(K) \geq C_d = 1/4d\log_2(20d)$. Denote by $F_1, \ldots, F_{2d}$ the external $d-1$-dimensional faces of L_i (namely, the faces of the cube Q_{2^i}). There are $2d$ faces, and each of them defines a convex set

$$K_i = \{x \in L_i : \forall j \neq i, d(x, F_i) \leq d(x, F_j)\}.$$

The sets $K_i \cap R$ are convex and one of them contain at least $(2d)^{-1}$ of the volume of $L_i \cap R$, which gives the desired K.

□

proof of Proposition 4.3.5. (i) Follows from Lemma 4.3.7.

(ii) $\#N_i = O(g(2^i))$, so there is a constant C_1 such that $\#N_i \leq C_1 g(2^i)$. By adding points to some of the N_i's we may assume that g gets integer values, and that $\#N_i = C_1 g(2^i)$ for every i. By adding more points, if needed, we may also assume that the function $\frac{g(x)}{x^d}$ is non-decreasing (observe that $g(x) = \Omega(x^d)$).

For a measurable set A we denote by $\mathfrak{D}(A) = \frac{|D \cap A|}{Vol(A)}$, the density of the set D in A, where $D = \bigcup_i N_i$. Note that for every $i > 0$ the layer L_i is the union of $4^d - 2^d$ cubes of edge length 2^{i-1}, that intersect only at their boundaries. So for every $i > 0$ we have

$$\mathfrak{D}(L_i) = \frac{C_1 g(2^i)}{(4^d - 2^d)(2^{i-1})^d} = \frac{C_1}{2^d - 1} \cdot \frac{g(2^i)}{2^i}.$$

Since $\frac{g(x)}{x^d}$ is non-decreasing, $\mathfrak{D}(L_i) \geq \mathfrak{D}(L_{i-1})$, and therefore $\mathfrak{D}(L_i) \geq \mathfrak{D}(Q_{2^{i-1}})$. Also note that for every $i > 0$ we have $Vol(L_i) = (2^d - 1) \cdot Vol(Q_{i-1})$, then

$$\#N_i = \mathfrak{D}(L_i) \cdot Vol(L_i) \geq \mathfrak{D}(Q_{2^{i-1}}) \cdot (2^d - 1) Vol(Q_{2^{i-1}}) = (2^d - 1) \#(D \cap Q_{2^{i-1}}).$$

In particular, for every i we have $\#(D \cap Q_{2^i}) \leq 2|N_i| = 2C_1 g(2^i)$. Then for a given n, let $i \in \mathbb{N}$ be such that $n \leq 2^i < 2n$. Then

$$\#(D \cap Q_n) \leq \#(D \cap Q_{2^i}) \leq 2C_1 g(2^i) \leq 2C_1 g(2n).$$

□

Proof of Proposition 4.3.1. We saw that (i) implies (ii) in Claim 4.3.2. For (ii) implies (i), let $\varepsilon_i = \alpha_d^{-1} C_d \cdot 2^{-di}$, where C_d is as in (4.2), and α_d is from Claim 4.2.4. Let N_i'' be and ε-net for $(X, \mathcal{S}_{\varepsilon_i})$ of size $O(g(\varepsilon_i^{-1/d}))$, and we may assume that $X = [-1, 1]^d$. Rescale the whole picture by a factor of 2^i in each axis. So X becomes Q_{2^i}, and N_i'' becomes $N_i' \subseteq Q_{2^i}$, a set of points of size $O(g(\varepsilon_i^{-1/d})) = O(g((\alpha_d^{-1} C_d)^{-1/d} \cdot 2^i))$ that intersect every box of volume $\varepsilon_i \cdot 2^{di} = \alpha_d^{-1} C_d$ in Q_{2^i}. Note that since $g(x)$ has polynomial growth we have $\#N_i' = O(g(2^i))$, and it follow from Claim 4.2.4 that N_i' intersects any convex set of volume C_d in Q_{2^i}. Let $N_i = N_i' \cap L_i$, where L_i is as in (4.1). So $\#N_i = O(g(2^i))$ and N_i intersects any convex set of volume C_d that is contained in L_i. Then by Proposition 4.3.5 we obtain a Danzer set $D = \bigcup_i N_i$ in $\mathbb{R}^d$ with growth rate $O(g(T))$. □

4.4 Proof of Theorem 4.1.5

To simplify the notation, we denote by $Q_n \subseteq \mathbb{R}^d$ the cube of edge length n centered at the origin in this section. We begin with the following proposition, which can be viewed as a special case of Theorem 4.2.10. For completeness, we add the proof of this proposition, that we learned from Saurabh Ray.

Proposition 4.4.1. *For any $n > 0$ there exists a $N \subseteq Q_n$ with $\#N = O(n^d \log n)$, where C depends only on d, such that for any box $R \subseteq Q_n$ of volume 1 we have $R \cap N \neq \varnothing$.*

Proof. Let Γ_n be a finite grid in Q_n that divides Q_n to cubes with side length $1/n$. Then each side of Q_n is divided into n^2 points, and therefore $\#\Gamma_n = n^{2d}$. Note that any box $R \subseteq Q_n$ of volume 1 that is contained in Q_n contains $\Omega(n^{2d}/n^d) = \Omega(n^d)$ points of Γ_n (up to an error of n^{d-1}), and at least $n^d/2$ points.

Let

$$p = \frac{c\log(n)}{n^d} \in (0,1),$$

where c depend only on d, and will be chosen later. Let N be a random subset of Γ_n that is obtained by choosing points from Γ_n randomly and independently with probability p. Then the $\#N$ is a binomial random variable $B(m = n^{2d}, p)$, where $m = n^{2d}$, with expectation

$$\mathbf{E}(\#N) = n^{2d} \cdot p = c \cdot n^d \log(n).$$

Since $\#N = B(m,p)$ the values of $\#N$ concentrate near $\mathbf{E}(\#N)$. To be precise, using Chernoff bound for example (see [C52]) one obtains

$$Prob[|\#N - \mathbf{E}(\#N)| \geq \mathbf{E}(\#N)/2] \leq e^{-\frac{\mathbf{E}(\#N)}{16}}.$$

So in particular with probability greater than $(n-1)/n$ we have

$$\frac{1}{2}cn^d \log(n) \leq \#N \leq \frac{3}{2}cn^d \log(n). \tag{4.4}$$

Such a set N misses a given box R if all the points in R were not taken to N. The probability for that event is at most

$$(1-p)^{n^d/2} = \left(1 - \frac{c\log(n)}{n^d}\right)^{n^d/2} \geq e^{-c\log(n)/2} = \frac{1}{n^{c/2}}.$$

Recall that we saw in the first example in 4.2.7 that $VCdim([0,1]^d, \{\text{closed half-spaces}\}) = d+1$. Since every box in $[0,1]^d$ is the intersection of $2d$ closed half-spaces, by Corollary 4.2.9 we have $VCdim([0,1]^d, \{\text{boxes}\}) \leq 4d(d+1)\log(2d(d+1)) \leq 8(d+1)^3 := q$ (we are even taking a subset of $\mathcal{S}_{2d}$, for $\mathcal{S} = \{\text{closed half-spaces}\}$). In particular, we also have

$$VCdim(Q_n, \mathcal{S} := \{R \cap \Gamma_n : R \subseteq Q_n \text{ is a box of volume } 1\}) \leq q.$$

By Lemma 4.2.8 we have

$$\#\mathcal{S} \leq \sum_{i=0}^{q} \binom{\#\Gamma_n}{i} \leq (q+1)n^{2dq}.$$

Pick $c/2 = 2dq+1 = O(d^4)$. So a standard union bound gives that the probability to miss some box R is at most:

$$(q+1)n^{2dq} \cdot \frac{1}{n^{c/2}} = O\left(\frac{1}{n}\right). \tag{4.5}$$

In particular, there is a set $N \subseteq \Gamma_n$ satisfying (4.4) and (4.5). That is a set N of size $O(n^d \log(n))$ that intersect every box of volume 1 in Q_n. $\square$

Proof of Theorem 4.1.5. Let $\varepsilon > 0$. Let $n \in \mathbb{N}$ be the minimal positive integer that satisfies $1/n^d \leq \varepsilon$. By Proposition 4.4.1 for every $n \in \mathbb{N}$ we have a set $N_n \subseteq Q_n$ of size $O(n^d \log(n))$ that intersect every box of volume 1 in Q_n. Rescale the whole picture by a factor of $1/n$ in each axis, we obtain a set $Y_n \subseteq [-1/2, 1/2]^d$ of size $O(n^d \log(n))$ that intersect every box of volume $1/n^d$ in $[-1/2, 1/2]^d$. In particular, we have an ε-net of size $O(n^d \log(n))$ for the range space $(X, \mathcal{S}_\varepsilon)$, where $X = [0,1]^d$ and $\mathcal{S}_\varepsilon =$ boxes of volume ε. Note that $n - 1 < \varepsilon^{-1/d} \leq n$, so we have showed (ii) of Proposition 4.3.1, with $g(x) = x^d \log(x)$, and therefore we have a Danzer set of growth rate $O(T^d \log(T))$. $\square$

4.5 Proof of Theorem 4.1.4

We now move to a discussion on non-examples for a Danzer set. Clearly, a lattice in $\mathbb{R}^d$ cannot be a Danzer set. In [BW71] Bambah and Woods use a nice dynamical argument to show that any finite union of grids (translated lattices) will also fail to be a Danzer set. We now show that other natural candidates also fail to be Danzer sets.

Our goal is to prove Theorem 4.1.4. Let H be a primitive substitution rule, with finitely many polygonal prototiles $\mathcal{F} = \{T_1, \ldots, T_n\}$ in $\mathbb{R}^d$, and inflation constant $\xi > 1$. Fix a tiling $\tau_0 \in X_H$. Recall that for a function $h : \mathcal{F} \to \bigcup \mathcal{F}$, with $h(T_i) \in T_i$ for every i, $Z_{\tau_0,h}$ denotes the separated net that is obtained by placing one point in each tile of τ_0, with respect to the choices of h. More precisely, each tile of τ_0 is equal to $T_i + v$, for some i, and some $v \in \mathbb{R}^d$. So the function h can be naturally extended to the collection of all tiles in τ_0. Then $Z_{\tau_0,h} = h(\tau_0)$. We shall keep using this notation later.

By a *polygonal tiling* we simply mean that the prototiles are d-dimensional polytopes, namely convex bounded sets that can be obtained as an intersection of finitely many half-spaces. Also recall Proposition 1.5.1, that tells us that for every $m \in \mathbb{N}$ there exists a tiling $\tau_m \in X_H$ with $(\xi H)^m(\tau_m) = \tau_0$. Denote by $\partial\tau$ the union of all the boundaries of tiles of a tiling τ, so we have $\partial\tau_j \subseteq \partial\tau_i$ for any $i \leq j$.

Remark 4.5.1. *The proof does not using the convexity of the tiles, but only the fact that one of the tiles contains a straight segment in its boundary. Therefore* *it can be extended to other cases where this property holds, for instance, if the prototiles are finite unions of cubes.*

Proof of Theorem 4.1.4. (i) Let $h : \mathcal{F} \to \bigcup \mathcal{F}$, with $h(T_i) \in T_i$ for every i. For a set A and $\delta > 0$ we denote by $U_\delta(A) = \{x \in \mathbb{R}^d : d(x, A) < \delta\}$.

We first consider the case where $h(T_i) \in int(T_i)$ for every i, where the main idea of the proof is presented. Denote by

$$\delta = \min_i \{d(h(T_i), \partial T_i)\}. \tag{4.6}$$

Notice that if a $d-1$-dimensional face of a tile in τ_0 contains a segment of length t, then the same type of tile in τ_m contains a segment of length $t \cdot \xi^m$ in one of its faces. The tiles are polygonal, so let m be large enough such that some face F of some tile in τ_m contains a segment L of length $\ell > \delta^{1-d}$. Since $\partial\tau_m \subseteq \partial\tau_0$, L is also contained in $\partial\tau_0$. By (4.6), $U_\delta(L)$ misses $Z_{\tau_0,h}$.

Let $v \in \mathbb{R}^d$ be a vector of length $\delta/2$ which is perpendicular to F, and let $L' = L + v$. So $U_{\delta/2}(L') \subseteq U_\delta(L)$, and $U_{\delta/2}(L')$ contains an open box R of volume $\ell \cdot \delta^{d-1} > 1$. So R contains a closed box R' of volume 1 that misses $Z_{\tau_0,h}$ (and also misses F).

Since R' also misses F, the proof of the general case is obtained by defining *delta* a bit differently. We leave it as an exercise to the reader to fill up the technical details here.

(ii) For every $i \in \{1, \ldots, n\}$ let μ_i be a given probability measure on T_i. For a set A and $\delta > 0$ denote by $V\delta(A) = \{x \in A : 0 < d(x, \partial A) < \delta\}$, so for every i there is some δ_i with $supp(\mu_i) \nsubseteq V_{\delta_i}(T_i)$. Then we may fix some $\delta > 0$ that satisfies $supp(\mu_i) \nsubseteq V_\delta(T_i)$ for every i.

Let $Y \subseteq \mathbb{R}^d$ be a random set that is obtained by choosing one point y_T in each tile T of τ_0 randomly and independently, with respect to the distributions μ_i on each prototile. So for every tile T of type i there is a positive probability $p_i > 0$ for the event $y_T \notin V_\delta(T)$. Denote by $p = \min_i p_i > 0$.

Fix $\ell > \delta^{1-d}$. It follows from the structure of substitution tiling that there are infinitely many segments L of length ℓ in $\partial\tau_0$, like we found in the proof of (i). It also follows from the structure of substitution tilings that there is a uniform bound $M \in \mathbb{N}$ to the number of tiles of τ_0 that such a segment L intersects. So given such a segment L, with probability at least $p^M > 0$ we have $y_T \notin V_\delta(T)$, for every tile $T \in \tau_0$ with $T \cap L \neq \varnothing$. Since there are infinitely many segments $L \subseteq \partial\tau_0$ of length ℓ, with probability 1 there is such a segment L with the above property. Like in the proof of (i), such a segment L gives rise to a box of volume 1 that misses Y.

□

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