A NOTE ON REDUCTION OF TILING PROBLEMS

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Abstract. We show that translational tiling problems in a quotient of $\mathbb{Z}^d$ can be effectively reduced or "simulated" by translational tiling problems in $\mathbb{Z}^d$. In particular, for any $d \in \mathbb{N}$, $k < d$ and $N_1, \ldots, N_k \in \mathbb{N}$ the existence of an aperiodic tile in $\mathbb{Z}^{d-k} \times (\mathbb{Z}/N_1\mathbb{Z} \times \ldots \times \mathbb{Z}/N_k\mathbb{Z})$ implies the existence of an aperiodic tile in $\mathbb{Z}^d$. Greenfeld and Tao have recently disproved the well-known periodic tiling conjecture in $\mathbb{Z}^d$ for sufficiently large $d \in \mathbb{N}$ by constructing an aperiodic tile in $\mathbb{Z}^{d-k} \times (\mathbb{Z}/N_1\mathbb{Z} \times \ldots \times \mathbb{Z}/N_k\mathbb{Z})$ for suitable $d, N_1, \ldots, N_k \in \mathbb{N}$.

1. Introduction

In this note, we consider translational tiles over finitely generated abelian groups. Any finitely generated abelian group is a quotient of $\mathbb{Z}^d$ for some positive integer d. Briefly, we show that for any tuple of finite subsets $F_1, \ldots, F_k$ in a quotient Γ of $\mathbb{Z}^d$, there exists an explicit tuple of finite subsets $\tilde{F}_1, \ldots, \tilde{F}_k$ of $\mathbb{Z}^d$ so that tilings of $\mathbb{Z}^d$ by $\tilde{F}_1, \ldots, \tilde{F}_k$ directly correspond, formally and explicitly, to tilings of Γ by $F_1, \ldots, F_k$. Informally, this means that tiling problems in $\mathbb{Z}^d$ can effectively simulate tiling problems in Γ.

The motivation for writing this note comes from the periodic tiling conjecture and questions about the decidability of the tiling problems. Bhattacharya [Bha20] proved that a finite subset $F \subset \mathbb{Z}^2$ can tile $\mathbb{Z}^2$ if and only if it admits a periodic tiling, resolving the $\mathbb{Z}^2$ case of the well-known periodic tiling conjecture. The periodic tiling conjecture seems to be currently open in $\mathbb{Z}^3$ and also in $\mathbb{Z}^2 \times \mathbb{Z}/N\mathbb{Z}$ for general $N \in \mathbb{N}$; see [GT21, subsection 1.3]. From our argument below, it follows that an affirmative solution of the periodic tiling conjecture in $\mathbb{Z}^3$ would automatically imply an affirmative resolution of the periodic tiling conjecture in $\mathbb{Z}^2 \times \mathbb{Z}/N\mathbb{Z}$ for all $N \in \mathbb{N}$.

The main result of [GT21] is the existence of a rank 2 finitely generated group Γ and two finite subsets $F_1, F_2 \subset \Gamma$ that can tile a certain periodic subset $E \subseteq \Gamma$, but such that no proof of the sentence "there is a tiling of E by F_1 and F_2" exists in ZFC. In particular, such sets F_1, F_2 cannot tile E periodically. Building on this construction, it was further shown in [GT21] that a similar statement holds when replacing Γ by $\mathbb{Z}^d$ for sufficiently large d. The reduction procedure described below is a slight elaboration of the construction applied by Greenfeld and Tao in [GT21]. The main new point is that the reduction procedure can be used in a general setting, independently of any properties of the given set of tiles, the number of tiles, the rank d of $\mathbb{Z}^d$, and the quotient group Γ of $\mathbb{Z}^d$.

Greenfeld and Tao [GT22] have recently disproved the periodic tiling conjecture in $\mathbb{Z}^d$ for sufficiently large $d \in \mathbb{N}$ by constructing an aperiodic tile in $\mathbb{Z}^{d-k} \times (\mathbb{Z}/N_1\mathbb{Z} \times \ldots \times \mathbb{Z}/N_k\mathbb{Z})$, for suitable $d, k, N_1, \ldots, N_k \in \mathbb{N}$.

We briefly describe our notational conventions that closely follow the conventions of [GT21]. We write $\biguplus_{i \in I} A_i$ to denote the union of pairwise disjoint sets. Given an abelian group Γ

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and subsets $A, B \subseteq \Gamma$, we write $A \oplus B = C$ to indicate that for every $c \in C$ there exists a unique pair $a \in A$ and $b \in B$ such that $c = a + b$. For $N \in \mathbb{N}$ and $A \subseteq \mathbb{Z}^d$, we denote $NA := \{Nv : \ v \in A\}$. Given finite subsets $F_1, \dots, F_k \subset \Gamma$ and $E \subseteq \Gamma$ we use the notation

$$\begin{aligned} &Tile(F_1, \dots, F_k; E) = \left\{ (A_1, \dots, A_k) \in (2^\Gamma)^k : \ \biguplus_{j=1}^{k} (F_j \oplus A_j) = E \right\}, \\ &\text{and} \\ &Tile_0(F_1, \dots, F_k; E) = \left\{ (A_1, \dots, A_k) \in Tile(F_1, \dots, F_k; E) : \ 0 \in \biguplus_{j=1}^{k} A_j \right\}. \end{aligned} \tag{1}$$

We say that a k-tuple $(A_1, \dots, A_k) \in (2^\Gamma)^k$ is *periodic* if there exist a finite index subgroup $\Gamma_0 < \Gamma$ that fixes every A_j. Equivalently, $(A_1, \dots, A_k) \in (2^\Gamma)^k$ is periodic if there exists finite sets $C_1, \dots, C_k \subset \Gamma$ such that $A_j = C_j \oplus \Gamma_0$ for all $1 \leq j \leq k$. A k-tuple $(F_1, \dots, F_k)$ of finite subsets of Γ is *aperiodic* if $Tile(F_1, \dots, F_k; \Gamma)$ is non-empty but does not contain a periodic k-tuple. Given a subgroup $L < \Gamma$ we say that $D \subseteq \Gamma$ is a *fundamental domain* for L if D contains exactly one representative of each coset of L. Equivalently, $D \subseteq \Gamma$ is a fundamental domain if $D \oplus L = \Gamma$.

Here is the statement of the main reduction result:

Theorem 1.1. *Let $d, k \in \mathbb{N}$, $\pi : \mathbb{Z}^d \to \Gamma$ a surjective group homomorphism and let $D \subseteq \mathbb{Z}^d$ be a fundamental domain for $\ker(\pi)$. Then there exists $N \in \mathbb{N}$ and finite subsets $T_1, \dots, T_{2k} \subset \mathbb{Z}^d$ such that for every $F_1, \dots, F_k$, finite subsets of Γ with $0 \in F_j$ for all $1 \leq j \leq k$, we have*

$$\begin{aligned} &Tile_0(\tilde{F}_1, \dots, \tilde{F}_k; \mathbb{Z}^d) = N\Big((\pi^{\otimes k})^{-1}\, Tile_0\left(F_1, \dots, F_k; \Gamma\right) \Big), \\ &\textit{where for } 1 \leq j \leq k \\ &\tilde{F}_j := \big(N(\pi^{-1}(F_j \setminus \{0\}) \cap D) \oplus T_j \big) \uplus T_{k+j} \quad \subset \mathbb{Z}^d. \end{aligned} \tag{2}$$

Here $\pi^{\otimes k} : (\mathbb{Z}^d)^k \to \Gamma^k$ is given by

$$\pi^{\otimes k}(v_1, \dots, v_k) = (\pi(v_1), \dots, \pi(v_k)) \textit{ for } v_1, \dots, v_k \in \mathbb{Z}^d.$$

We state some direct corollaries of Theorem 1.1. Suppose $\pi : \mathbb{Z}^d \to \Gamma$ is a surjective group homomorphism.

Corollary 1.2. *If there exists an aperiodic k-tuple for Γ, then there exists an aperiodic k-tuple for $\mathbb{Z}^d$.*

Corollary 1.3. *For every $k \in \mathbb{N}$, if tiling by k tiles in $\mathbb{Z}^d$ is algorithmically decidable, then tiling by k tiles in Γ is algorithmically decidable.*

Corollary 1.4. *Let $F_1, \dots, F_k$ be finite subsets of Γ and let $\mathcal{T}$ be a theory in first-order logic in which the proof of (2) can be expressed. If the sentence $Tile(\tilde{F}_1, \dots, \tilde{F}_k; \mathbb{Z}^d) \neq \emptyset$ is provable in $\mathcal{T}$, then the sentence $Tile(F_1, \dots, F_k; \Gamma) \neq \emptyset$ is provable in $\mathcal{T}$.*

Readers familiar with logic can easily convince themselves that the proof of (2) can be expressed in ZFC, for any explicit finitely generated abelian group Γ (in the sense of [GT21]) and any explicit homomorphism $\pi : \mathbb{Z}^d \to \Gamma$.

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2. Proof of the reduction theorem

The proof of Theorem 1.1 is based on the following lemma. The case $k = 1$ and $L = \{0\}$ is essentially Lemma 9.3 of [GT21]), where the basic idea has been attributed to Golomb [Gol70].

Lemma 2.1 (Rigid tuple of tiles). *Let $d, s \in \mathbb{N}$, let $L < \mathbb{Z}^d$, and let $D \subseteq \mathbb{Z}^d$ be a fundamental domain for L. Then there exists $N \in \mathbb{N}$ and finite sets $T, T_1, \dots, T_s \subset \mathbb{Z}^d$, with $0 \in T$ and $0 \in T_j$ for every $1 \leq j \leq s$, such that*

(a) $\mathit{Tile}_0(T; \mathbb{Z}^d) = \{N\mathbb{Z}^d\}$.

(b) For every $1 \leq j \leq s$ we have $T_j \oplus NL = T \oplus NL$.

(c) $(\tilde{C}_1, \dots, \tilde{C}_s) \in \mathit{Tile}_0(T_1, \dots, T_s; \mathbb{Z}^d)$ if and only if $\biguplus_{j=1}^{s} \tilde{C}_j = N\mathbb{Z}^d$ and each $\tilde{C}_j$ is NL-periodic.

Remark 2.2.

(i) The tiles $T, T_1, \dots, T_s$ that are defined in the proof below are essentially boxes of side length N, with bumps and dents, such as pieces of a jigsaw puzzle, whose purpose is to enforce their positions (see Figure 1). For simplicity, the bumps and dents in the proof are of the form of "a frame" of width 1, i.e. the difference between two centered boxes, $B_n \setminus B_{n-1}$. We did not attempt to minimize the number N.

(ii) We note that as a consequence of (c), if $(\tilde{C}_1, \dots, \tilde{C}_s) \in \mathit{Tile}_0(T_1, \dots, T_s; \mathbb{Z}^d)$ then any permutation of $\{\tilde{C}_1, \dots, \tilde{C}_s\}$ is also in $\mathit{Tile}_0(T_1, \dots, T_s; \mathbb{Z}^d)$.

Proof. We prove the claim only for $d \geq 2$, since the proof for the case $d = 1$ requires slightly different considerations but is not more difficult.

Let $d, s \in \mathbb{N}$ and $L < \mathbb{Z}^d$ be given. Since L is a finitely generated abelian group of rank at most d, there exists $r \leq d$ and $w_1, \dots, w_r \in L$ that generate L as a free abelian group in the sense that $L = \mathbb{Z}w_1 \oplus \ldots \oplus \mathbb{Z}w_r$. Also, let $e_1, \dots, e_d \in \mathbb{Z}^d$ be a set of free generators for $\mathbb{Z}^d$ (say, the standard generators), and for $n \geq 0$, let

$$B_n := \{-n, \dots, n\}^d \subset \mathbb{Z}^d$$

and

$$S_n := B_n \setminus B_{n-1} \text{ for } n \geq 1.$$

The S_i's play the role of the bumps and dents in our tiles, and the properties that we need from them are that they are all bounded and that for every $i \neq j$ the set S_i does not contain a translated copy of the set S_j.

Choose $m \in \mathbb{N}$ large enough so that B_m contains disjoint translated copies of all the boxes B_l for $1 \leq l \leq d + rs$ (e.g. $m \geq (d + rs)^2$), and let $N = 2m + 1$.

Choose translation vectors $v_1, \dots, v_{d+rs} \in B_m$ so that

$$\forall 1 \leq i \leq d + rs : \quad (v_i + B_{d+rs}) \subseteq B_m,$$

and

$$\forall 1 \leq i_1 < i_2 \leq d + rs : \quad (v_{i_1} + B_{d+rs}) \cap (v_{i_2} + B_{d+rs}) = \emptyset.$$

We first define the tile $T \subset \mathbb{Z}^d$ as follows:

$$T := \left(B_m \setminus \biguplus_{i=1}^{d} (v_i + S_i) \right) \uplus \biguplus_{i=1}^{d} (Ne_i + v_i + S_i)\,. \tag{3}$$

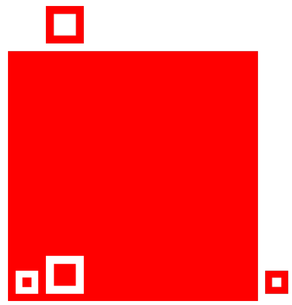

FIGURE 1. For a specific choice of vectors v_i's, the tile T from (3) is illustrated here for $d = 2$.

We refer to the missing translates of S_i's in T as *dents*, and to the disconnected translates of the S_i's as *bumps*. We claim that $Tile_0(T; \mathbb{Z}^d) = \{N\mathbb{Z}^d\}$. The claim is fairly evident from Figure 1. We provide some details below.

The properties of T that we use, which are straightforward from (3), are the following: Firstly, T is a fundamental domain for $N\mathbb{Z}^d$. Secondly, for every $1 \leq i \leq d$ the only $v \in T$ such that $v + S_i \cap T = \emptyset$ is v_i. Thus, $Ne_i + T$ is the unique translate of T that can cover Ne_i without intersecting the corresponding bump $Ne_i + v_i + S_i$. See Figure 1. Since T is a fundamental domain for $N\mathbb{Z}^d$, we have $N\mathbb{Z}^d \in Tile_0(T; \mathbb{Z}^d)$. Now suppose $C \in Tile_0(T; \mathbb{Z}^d)$, then for every $v \in C$ and $1 \leq i \leq d$ we also have $v \pm Ne_i \in C$, because each bump of $v + T$ must fit into a corresponding dent and vice versa. So C must be a union of cosets of $N\mathbb{Z}^d$. But since T is a fundamental domain for $N\mathbb{Z}^d$, it follows that C must be a coset of $N\mathbb{Z}^d$. Since $0 \in C$ we must have $C = N\mathbb{Z}^d$.

We now define the tiles $T_1, \dots, T_s$. For $1 \leq j \leq s$ the tile T_j is defined by adding additional r bumps and r dents to the tile T. The bumps and dents are of the form S_l, for $d + r(j-1) < l \leq d + jr$:

$$T_j := \left(T \setminus \biguplus_{l=1}^{r} \left(v_{d+r(j-1)+l} + S_{d+r(j-1)+l} \right) \right) \uplus \biguplus_{l=1}^{r} \left(Nw_l + v_{d+r(j-1)+l} + S_{d+r(j-1)+l} \right). \quad (4)$$

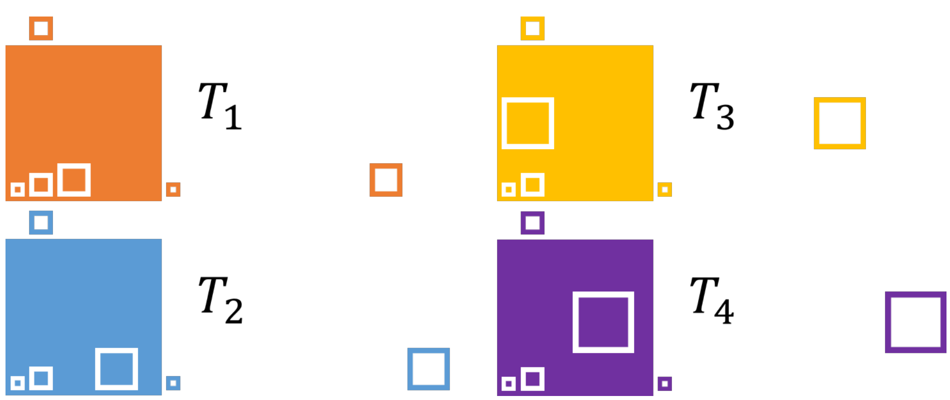

FIGURE 2. With the same parameters as in Figure 1, the tiles T_j for $j \in \{1, 2, 3, 4\}$, from (4), are illustrated above. Here $s = 4$, L is of rank $r = 1$ and $w_1 = (2, 0)$.

Direct inspection of (4) shows that for any $1 \le j \le s$ we have

$$T_j \oplus NL = T \oplus NL \tag{5}$$

From (5) and $T \oplus N\mathbb{Z}^d = \mathbb{Z}^d$ it follows that whenever $\{\tilde{C}_1, \dots, \tilde{C}_s\}$ is a partition of $N\mathbb{Z}^d$ into NL-periodic sets, we have $(\tilde{C}_1, \dots, \tilde{C}_s) \in \mathit{Tile}_0(T_1, \dots, T_s; \mathbb{Z}^d)$.

Now we show that for any $(\tilde{C}_1, \dots, \tilde{C}_s) \in \mathit{Tile}_0(T_1, \dots, T_s; \mathbb{Z}^d)$ we have that $\{\tilde{C}_1, \dots, \tilde{C}_s\}$ is a partition of $N\mathbb{Z}^d$ into NL-periodic sets. As soon as we show that each $\tilde{C}_j$ is NL-periodic, it will follow from (5) that $\tilde{C}_j \oplus T_j = \tilde{C}_j \oplus T$ and so $\biguplus_{j=1}^s \tilde{C}_j \in \mathit{Tile}_0(T; \mathbb{Z}^d)$, thus $\biguplus_{j=1}^s \tilde{C}_j = N\mathbb{Z}^d$. So it is left to explain why each $\tilde{C}_j$ is NL-periodic. The argument is essentially the same as the argument showing that each $C \in \mathit{Tile}_0(T; \mathbb{Z}^d)$ is $N\mathbb{Z}^d$-periodic, and should be fairly clear from Figure 2. In view of (4), the additional property of the collection $T_1, \dots, T_s$ that is needed here is the following: For every $1 \le l \le r$ and every $1 \le j, j' \le s$, if $v + T_{j'}$ covers $v_{d+r(j-1)+l}$ without intersecting the bump $Nw_l + v_{d+r(j-1)+l} + S_{d+r(j-1)+l}$ of T_j, then we must have $j = j'$ and $v = Nw_l$. See Figure 2. $\square$

Figure 3 illustrates the construction of the tiles $\tilde{F}_j \subset \mathbb{Z}^d$, as defined in (2).

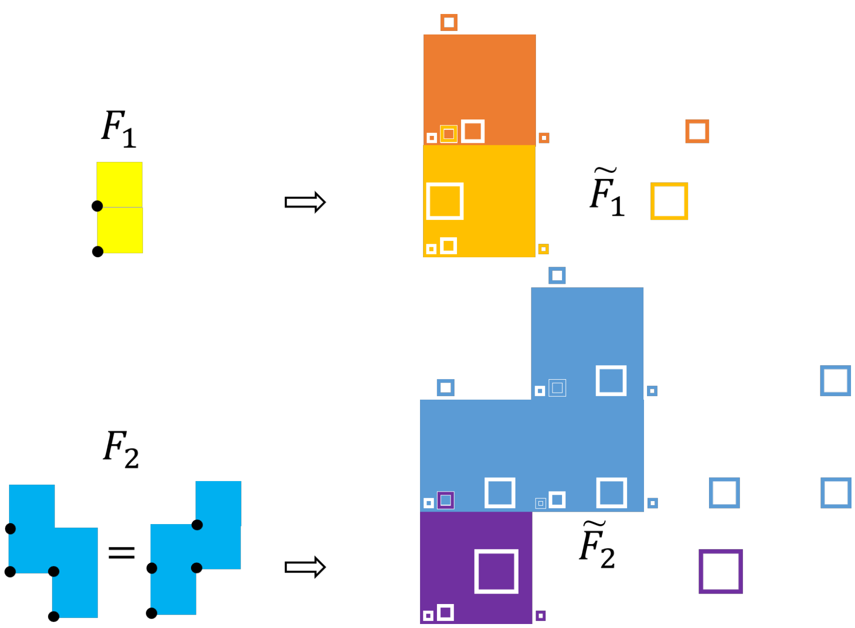

FIGURE 3. For $d = 2$, $s = 4$ and $\Gamma = (\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}$, the figure illustrates a particular choice of tiles F_1, F_2 that can tile the "vertical strip" Γ, and the corresponding construction of $\tilde{F}_1$ and $\tilde{F}_2$, using T_1, T_2, T_3 and T_4 from Figure 2. The T_j's are constructed in this case using $L = \ker(\pi) = \langle (2, 0) \rangle$. Note the the two pictures of F_2 represent the same subset of Γ.

For convenience we state and proof the following elementary lemma:

Lemma 2.3. *Let $\pi : \mathbb{Z}^d \to \Gamma$ be a surjective group homomorphism, and let $L = \ker(\pi) < \mathbb{Z}^d$.*

(a) If $A, B, C \subseteq \mathbb{Z}^d$ satisfy that $A \oplus B = C$ and B is L-periodic then $\pi(A) \oplus \pi(B) = \pi(C)$.

(b) If $C_1, \dots, C_k \subseteq \mathbb{Z}^d$ are each L-periodic and $\biguplus_{j=1}^k C_j = C$ then $\biguplus_{j=1}^k \pi(C_j) = \pi(C)$.

Proof. (a) The equality $\pi(A) + \pi(B) = \pi(C)$ follows directly because π is a group homomorphism. We need to show that the sum is direct, namely that the representation $\pi(c) = \pi(a) + \pi(b)$ with $a \in A$, $b \in B$ is unique for every $c \in C$. Suppose $\pi(a_1)+\pi(b_1) = \pi(a_2)+\pi(b_2)$. Since every $c \in C$ has a unique representation $c = a+b$ with $a \in A$ and $b \in B$, we conclude that

$$v := (a_1 + b_1) - (a_2 + b_2) \in L.$$

Set $b_3 := b_2 + v$. By the assumption that B is L-periodic, it follows that $b_3 \in B$. Then $a_2 + b_3 = a_1 + b_1$ are two representations of the same element as a sum of an element of A and an element of B. It follows that $a_1 = a_2$ and $b_1 = b_3$, so $b_1 = b_2 + v$. This means that $\pi(a_1) = \pi(a_2)$ and $\pi(b_1) = \pi(b_2)$. We have thus proved that the representation $\pi(c) = \pi(a) + \pi(b)$ with $a \in A$, $b \in B$ is unique for every $c \in C$.

(b) Using induction, it is enough to prove the case $k = 2$. Clearly, $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$ for every function f and sets C_1, C_2 in its domain, so we only need to show that $\pi(C_1) \cap \pi(C_2) = \emptyset$ under the assumptions that $C_1 \cap C_2 = \emptyset$ and that each C_i is L-periodic. Otherwise, there are $c_1 \in C_1$ and $c_2 \in C_2$ such that $\pi(c_1) = \pi(c_2)$. This means that $c_1 - c_2 \in L$. But C_1 is L-periodic and $c_1 \in C_1$ so $c_2 \in C_1$, contradicting $C_1 \cap C_2 = \emptyset$.

□

We are now ready to prove our main result.

Proof of Theorem 1.1. Given $d, k \in \mathbb{N}$ and a surjective homomorphism $\pi : \mathbb{Z}^d \to \Gamma$ let $L = \ker(\pi)$, let D be a fundamental domain for L that contains 0, and let $T, T_1, \ldots, T_{2k} \subseteq \mathbb{Z}^d$ and $N \in \mathbb{N}$ satisfy the conclusion of Lemma 2.1 with respect to $d, s = 2k$ and L.

Given finite sets $F_1, \ldots, F_k \subset \Gamma$, let $\tilde{F}_1, \ldots, \tilde{F}_k \subset \mathbb{Z}^d$ be as in (2). For the first inclusion, suppose $(A_1, \ldots, A_k) \in \mathit{Tile}_0(F_1, \ldots, F_k; \Gamma)$, then by definition $\biguplus_{j=1}^{k}(F_j \oplus A_j) = \Gamma$. Applying π^{-1}, we obtain that

$$\biguplus_{j=1}^{k} \pi^{-1}(F_j \oplus A_j) = \mathbb{Z}^d. \tag{6}$$

For $1 \le j \le k$ we define

$$\tilde{C}_j := \big((N\pi^{-1}(F_j \setminus \{0\}) \cap D) \oplus N\pi^{-1}(A_j)\big) \text{ and } \tilde{C}_{j+k} := N\pi^{-1}(A_j). \tag{7}$$

By the identity

$$\pi^{-1}(F_j \oplus A_j) = (\pi^{-1}(F_j) \cap D) \oplus \pi^{-1}(A_j) = (\pi^{-1}(F_j \setminus \{0\}) \cap D) \oplus \pi^{-1}(A_j) \uplus \pi^{-1}(A_j),$$

and in view of (7), we see that for every $1 \le j \le k$ we have $\pi^{-1}(F_j \oplus A_j) = \frac{1}{N}\left[\tilde{C}_j \uplus \tilde{C}_{j+k}\right]$. Thus multiplying the identity in (6) by N yields $\biguplus_{j=1}^{2k} \tilde{C}_j = N\mathbb{Z}^d$.

Note that $N\pi^{-1}(A_j)$ is NL-periodic, thus $\{\tilde{C}_1, \ldots, \tilde{C}_{2k}\}$ is a partition of $N\mathbb{Z}^d$ into NL-periodic sets. By Lemma 2.1 it follows that

$$\left(\tilde{C}_1, \ldots, \tilde{C}_{2k}\right) \in \mathit{Tile}_0(T_1, \ldots, T_{2k}; \mathbb{Z}^d).$$

Using (7), the expression $\biguplus_{j=1}^{2k} T_j \oplus \tilde{C}_j = \mathbb{Z}^d$ can be rearranged to

$$\biguplus_{j=1}^{k} \left[\big((N\pi^{-1}(F_j \setminus \{0\}) \cap D) \oplus T_j\big) \uplus T_{j+k}\right] \oplus N\pi^{-1}(A_j) = \mathbb{Z}^d.$$

But in view of the definition of the $\tilde{F}_j$ in (2), this means that $(N\pi^{-1}(A_1), \ldots, N\pi^{-1}(A_k)) \in Tile_0(\tilde{F}_1, \ldots, \tilde{F}_k; \mathbb{Z}^d)$. We conclude that

$$N\Big((\pi^{\otimes k})^{-1}\, Tile_0\,(F_1, \ldots, F_k; \Gamma)\Big) \subseteq Tile_0(\tilde{F}_1, \ldots, \tilde{F}_k; \mathbb{Z}^d).$$

To see the other inclusion, let $(\tilde{A}_1, \ldots, \tilde{A}_k) \in Tile_0(\tilde{F}_1, \ldots, \tilde{F}_k; \mathbb{Z}^d)$. Then the following statements hold:

(1) $0 \in \biguplus_{j=1}^k \tilde{A}_j$.
(2) $\biguplus_{j=1}^k \tilde{F}_j \oplus \tilde{A}_j = \mathbb{Z}^d$.

The second equation can be rewritten as follows:

$$\biguplus_{j=1}^{k} \left[\big((N\pi^{-1}(F_j \setminus \{0\}) \cap D) \oplus T_j\big) \uplus T_{j+k}\right] \oplus \tilde{A}_j = \mathbb{Z}^d.$$

This can be rearranged as:

$$\biguplus_{j=1}^{k} \Big(N(\pi^{-1}(F_j \setminus \{0\}) \cap D) \oplus T_j \oplus \tilde{A}_j\Big) \uplus \biguplus_{j=1}^{k} (T_{j+k} \oplus \tilde{A}_j) = \mathbb{Z}^d. \tag{8}$$

Thus $(\tilde{A}_1, \ldots, \tilde{A}_k) \in Tile_0(\tilde{F}_1, \ldots, \tilde{F}_k; \mathbb{Z}^d)$ if and only if

$$\Big(\tilde{C}_1, \ldots, \tilde{C}_{2k}\Big) \in Tile_0(T_1, \ldots, T_{2k}; \mathbb{Z}^d),$$

where

$$\tilde{C}_j := N(\pi^{-1}(F_j \setminus \{0\}) \cap D) \oplus \tilde{A}_j \text{ and } \tilde{C}_{j+k} := \tilde{A}_j \quad \text{for } 1 \le j \le k.$$

By Lemma 2.1, this happens if and only if $\{\tilde{C}_1, \ldots, \tilde{C}_{2k}\}$ is a partition of $N\mathbb{Z}^d$ into NL-periodic sets. Since $\tilde{C}_j \subseteq N\mathbb{Z}^d$ it follows that $\tilde{A}_j \subseteq N\mathbb{Z}^d$. Now because $\tilde{C}_{j+k}$ is NL-periodic, it follows that each $\tilde{A}_j$ is NL-periodic. This means that for every $1 \le j \le k$ there exists $A_j \subseteq \Gamma$ such that

$$\tilde{A}_j = N\pi^{-1}(A_j). \tag{9}$$

It remains to show that $(A_1, \ldots, A_k) \in Tile_0(F_1, \ldots, F_k; \Gamma)$. Because $0 \in \biguplus_{j=1}^k \tilde{A}_j$, it follows that $0 \in \biguplus_{j=1}^k A_j$. Using the property $T_l \oplus NL = T \oplus NL$ for every $1 \le l \le 2k$, and the fact that each $\tilde{A}_j$ is NL-periodic, we may replace every T_l in (8) by T to obtain:

$$\biguplus_{j=1}^{k} \Big(N(\pi^{-1}(F_j \setminus \{0\}) \cap D) \oplus T \oplus \tilde{A}_j\Big) \uplus \biguplus_{j=1}^{k} (T \oplus \tilde{A}_j) = \mathbb{Z}^d.$$

Using the fact that $0 \in D$, the left-hand side can be rearranged to obtain the following:

$$\left[\biguplus_{j=1}^{k} \Big(N(\pi^{-1}(F_j) \cap D) \oplus \tilde{A}_j\Big)\right] \oplus T = \mathbb{Z}^d.$$

Recall that by Lemma 2.1 we have $Tile_0(T; \mathbb{Z}^d) = \{N\mathbb{Z}^d\}$, thus

$$\biguplus_{j=1}^{k} \Big(N(\pi^{-1}(F_j) \cap D) \oplus \tilde{A}_j\Big) = N\mathbb{Z}^d.$$

Plugging in (9) we conclude that $\biguplus_{j=1}^{k}\left(N(\pi^{-1}(F_j)\cap D)\oplus N\pi^{-1}(A_j)\right)=N\mathbb{Z}^d$. Since $v\mapsto Nv$ induces a group isomorphism between $\mathbb{Z}^d$ and $N\mathbb{Z}^d$ we get $\biguplus_{j=1}^{k}\left((\pi^{-1}(F_j)\cap D)\oplus\pi^{-1}(A_j)\right)=\mathbb{Z}^d$. By Lemma 2.3, using the fact that $\pi^{-1}(A_j)$ is L-periodic, it follows that $\biguplus_{j=1}^{k}F_j\oplus A_j=\Gamma$. We have thus shown that

$$Tile_0(\tilde{F}_1,\ldots,\tilde{F}_k;\mathbb{Z}^d)\subseteq N\left((\pi^{\otimes k})^{-1}\,Tile_0\left(F_1,\ldots,F_k;\Gamma\right)\right).$$

□

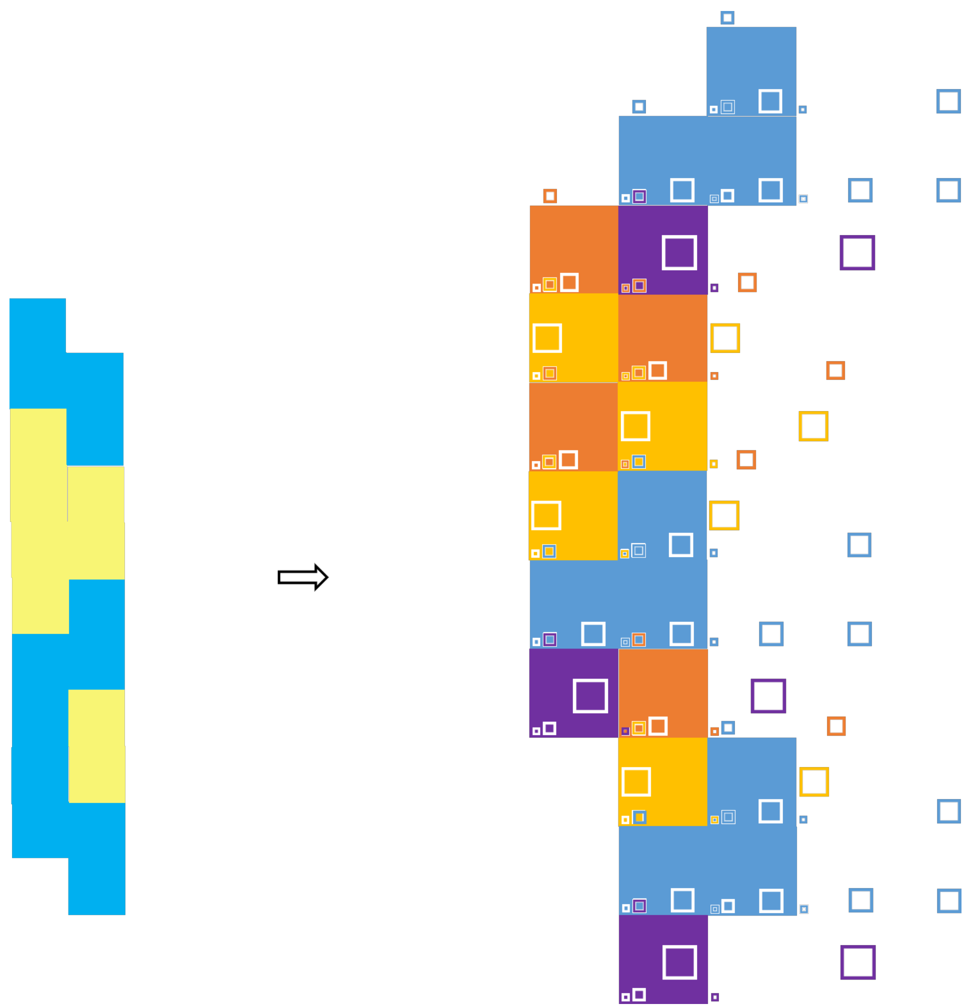

FIGURE 4. A patch of Γ-tiling using F_1 and F_2 and a patch of the corresponding $\mathbb{Z}^2$ -tiling using $\tilde{F}_1$ and F_2.

3. Corollaries regarding periodicity, topological conjugacy, algorithmic decidability and more

Proof of Corollary 1.2. Suppose $(F_1,\ldots,F_k)$ is an aperiodic k-tuple of finite sets in Γ, then $Tile(F_1,\ldots,F_k;\Gamma)\neq\emptyset$. By (2), we have that $Tile(\tilde{F}_1,\ldots,\tilde{F}_k;\mathbb{Z}^d)\neq\emptyset$. If we assume that $(\tilde{F}_1,\ldots,\tilde{F}_k)$ is not aperiodic, then by (2) there exists $(A_1,\ldots,A_k)\in Tile(F_1,\ldots,F_k;\Gamma)$ such that $(N\pi^{-1}(A_1),\ldots,N\pi^{-1}(A_k))$ is a periodic k tuple in $\mathbb{Z}^d$. But this implies that $(A_1,\ldots,A_k)$

is periodic. Thus, the existence of an aperiodic k-tuple in $\mathbb{Z}^d$ implies the existence of an aperiodic k-tuple in Γ. $\square$

Proof of Corollary 1.3. Assume that for a particular $k \in \mathbb{N}$ a tiling by k tiles in $\mathbb{Z}^d$ is algorithmically decidable. By definition, this means that there exists a Turing machine $\tilde{M}$ that takes as input a k-tuple $\tilde{F}_1, \dots, \tilde{F}_k$ of finite subsets in $\mathbb{Z}^d$ and accepts the input if and only if $Tile(\tilde{F}_1, \dots, \tilde{F}_k; \mathbb{Z}^d) \neq \emptyset$. The Turing machine $\tilde{M}$ always halts in finite time. Then we can construct a Turing machine M that takes as input a k-tuple $F_1, \dots, F_k$ of finite subsets in Γ, produces the finite subsets $\tilde{F}_1, \dots, \tilde{F}_k$ of $\mathbb{Z}^d$ described in the statement of Theorem 1.1, then runs $\tilde{M}(\tilde{F}_1, \dots, \tilde{F}_k)$. Clearly M halts in finite time on input $F_1, \dots, F_k$ if and only if $\tilde{M}$ halts with input $\tilde{F}_1, \dots, \tilde{F}_k$. The Turing machine M accepts $F_1, \dots, F_k$ if and only if $\tilde{M}$ accepts $\tilde{F}_1, \dots, \tilde{F}_k$, which by assumption happens if and only if $Tile(\tilde{F}_1, \dots, \tilde{F}_k; \mathbb{Z}^d) \neq \emptyset$. By Theorem 1.1 this happens if and only if $Tile(F_1, \dots, F_k; \mathbb{Z}^d) \neq \emptyset$. $\square$

Proof of Corollary 1.4. The concatenation of the proof of $Tile(\tilde{F}_1, \dots, \tilde{F}_k; \mathbb{Z}^d) \neq \emptyset$ in $\mathcal{T}$ with the proof of (2) in $\mathcal{T}$ yields a proof of $Tile(F_1, \dots, F_k; \Gamma) \neq \emptyset$ in $\mathcal{T}$. $\square$

To conclude this note, we explain why Corollary 1.2 is a consequence of a stronger correspondence between translational tilings in Γ and in $\mathbb{Z}^d$, expressed in terms of topological dynamics.

A $\mathbb{Z}^d$-*topological dynamical system* is a pair (X, T) where X is a compact topological space and $T \in Hom(\mathbb{Z}^d, Homeo(X))$, the group of homomorphisms from $\mathbb{Z}^d$ to the group $Homeo(X)$ of self-homeomorphisms of X. We say that $\mathbb{Z}^d$-topological dynamical systems (X, T) and (Y, S) are *topologically conjugate* or *isomorphic* if there exists a homeomorphism $\Phi : X \to Y$ such that $S_v = \Phi \circ T_v \circ \Phi^{-1}$ for every $v \in \mathbb{Z}^d$.

The space $(2^\Gamma)^k$ of k-tuples of finite subsets of Γ, equipped with the product topology, is a totally disconnected compact metrizable space. Note that that wherever $\pi : \mathbb{Z}^d \to \Gamma$ is a surjective group homomorphism, the group $\mathbb{Z}^d$ acts on $(2^\Gamma)^k$ by homeomorphism via $\sigma^{(\pi)} \in Hom(\mathbb{Z}^d, Homeo(2^\Gamma)^k)$, where for every $v \in \mathbb{Z}^d$, $\sigma_v^{(\pi)} \in Homeo((2^\Gamma)^k)$ is given by

$$\sigma_v^{(\pi)}(A_1, \dots, A_k) = (A_1 + \pi(v), \dots, A_k + \pi(v)) \text{ for } (A_1, \dots, A_k) \in (2^\Gamma)^k.$$

For any k-tuple $(F_1, \dots, F_k)$ of finite sets the space $Tile(F_1, \dots, F_k; \Gamma)$ is a closed, $\sigma^{(\pi)}$-invariant subset of $(2^\Gamma)^k$. So $(Tile(F_1, \dots, F_k; \Gamma), \sigma^{(\pi)})$ is a $\mathbb{Z}^d$-topological dynamical system. Let $N \in \mathbb{N}$ and $(\tilde{F}_1, \dots, \tilde{F}_k)$ be a k-tuple of finite subsets of $\mathbb{Z}^d$ that satisfies (2). Let $\sigma, \sigma^{(N)} \in Hom(\mathbb{Z}^d, Homeo((2^{\mathbb{Z}^d})^k))$ be given by

$$\sigma_v(A_1, \dots, A_k) = (A_1 + v, \dots, A_k + v) \text{ for } (A_1, \dots, A_k) \in (2^{\mathbb{Z}^d})^k.$$

and

$$\sigma_v^{(N)}(A_1, \dots, A_k) = (A_1 + Nv, \dots, A_k + Nv) \text{ for } (A_1, \dots, A_k) \in (2^{\mathbb{Z}^d})^k.$$

Let $D_N := \{0, \dots, N-1\}^d \subset \mathbb{Z}^d$. For each $t \in D_N$ let

$$X_t = \bigcup_{v \in N\mathbb{Z}^d + t} \sigma_v \left(Tile_0(\tilde{F}_1, \dots, \tilde{F}_k; \mathbb{Z}^d) \right).$$

Then each X_t is a closed, $\sigma^{(N)}$-invariant subset of $Tile(\tilde{F}_1, \dots, \tilde{F}_k; \mathbb{Z}^d)$, and $Tile(\tilde{F}_1, \dots, \tilde{F}_k; \mathbb{Z}^d) = \biguplus_{t \in D_N} X_t$. For each $t \in D_N$, the $\mathbb{Z}^d$-topological dynamical system $(X_t, \sigma^{(N)})$ is topologically conjugate to $(Tile(F_1,\dots,F_k;\Gamma),\sigma^{(\pi)})$ via the map

$$(A_1,\dots,A_k)\mapsto\left(\pi\left(\frac{1}{N}(A_1-t)\right),\dots,\pi\left(\frac{1}{N}(A_k-t)\right)\right).$$

We conclude:

Corollary 3.1. *For every k-tuple of finite subsets of $(F_1,\dots,F_k)$ of Γ there exists a k-tuple $(\tilde{F}_1,\dots,\tilde{F}_k)$ of finite subsets of $\mathbb{Z}^d$ and $N\in\mathbb{N}$ such that the $\mathbb{Z}^d$-topological dynamical system $(Tile(\tilde{F}_1,\dots,\tilde{F}_k;\mathbb{Z}^d),\sigma^{(N)})$ is topologically conjugate to the $\mathbb{Z}^d$-topological dynamical system $(Tile(F_1,\dots,F_k;\Gamma)\times\{1,\dots,N^d\},\sigma^{(\pi)}\times Id)$.*

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