MEASURE THEORY

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Contents

Measure Theory

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Lecture 1: Introduction

1.1. Measuring things

Already the ancient Greeks developed a theory of how to measure length, area, and volume and area of $1, 2$ and 3 dimensional objects. In this setting (*i.e.* in $\mathbb{R}^d$ for $d \leq 3$) it stands to reason that the "size" or "measure" of an object must satisfy some basic axioms:

- If $m(A)$ is the measure of a set A, it should be the same for any reflection, translation or rotation of A. That is, $m(A) = m(A + x) = m(UA)$ where U is a rotation matrix.

  For example, $[0, 1]^2$ should heave the same measure as $[-2, -1]^2$ which should have the same measure as $\left\{(x, y) \ : \ |x| + |y| \leq \frac{1}{\sqrt{2}}\right\}$, the diamond of side-length 1.
- If A can be broken into disjoint pieces, the sum of their measures should be the measure of A. That is, $m(A \uplus B) = m(A) + m(B)$.

  Example: $[0, 1] \times [0, 2]$ should have measure that is the sum of the measures of $[0, 1]^2$ and $[0, 1] \times (1, 2]$.

✓ We use $\uplus$ to denote disjoint union; that is, $A \uplus B$ is not only notation for a set, but this notation claims that $A \cap B = \emptyset$. The small + sign remind us of the additive property above.

This is already quite fruitful. If the unit square $(0, 1]^2$ is of measure 1, then:

- We can determine the measure of any square of rational side length.

$$m((0, n]^2) = \sum_{j,k=1}^{n} m((j-1, j] \times (k-1, k]) = n^2$$

and

$$m((0,1]^2) = \sum_{j,k=1}^{n} m((\tfrac{j-1}{n}, \tfrac{j}{n}] \times (\tfrac{k-1}{n}, \tfrac{k}{n}]) = n^2 m((0, \tfrac{1}{n}]^2).$$

- We can measure any rectangle of rational side length: decompose it into squares.
- We can measure right-angle triangles: disjoint union of two is a rectangle.
- We can then measure any triangle, by bounding it in a rectangle and subtracting the excess right-angle triangles.
- With triangles we can measure any polygon.

✓ What about measuring a disc?

1.2. Elementary measure

Let us first formally define the above.

• **Definition 1.1** (Boxes)**.** Consider $\mathbb{R}^d$ for some $d \geq 1$.

- $I \subset \mathbb{R}$ is an **interval** if I is one of $[a,b], (a,b), (a,b], [a,b)$ for some $-\infty < a \leq b < \infty$. Note that we allow the singleton $\{a\} = [a,a]$ as an interval, and the empty set $\emptyset = (a,a)$. The **length** or **measure** of such I is defined to be $|I| = m(I) = b - a$.
- $B \subset \mathbb{R}^d$ is a **box** if $B = I_1 \times \cdots \times I_d$ where I_j are intervals. The **volume** or **measure** of such a box B is defined to be $|B| = m(B) = |I_1| \cdots |I_d|$.
- $E \subset \mathbb{R}^d$ is an **elementary set** if $E = B_1 \cup \cdots \cup B_n$ for some finite number of boxes.
- $\mathcal{E}_0 = \mathcal{E}_0(\mathbb{R}^d)$ denotes the set of elementary sets in $\mathbb{R}^d$.

► **Exercise 1.1.** Show that $\mathcal{E}_0$ is closed under finite unions, finite intersections, set-difference, symmetric difference and translations. That is, show that if E, F are elementary sets then so are:

- $E \cup F$ and $E \cap F$,
- $E \setminus F := \{x \in E \ : \ x \notin F\} = E \cap F^c$,
- $E \triangle F := (E \setminus F) \cup (F \setminus E)$,

- $E + x := \{y + x \ : \ y \in E\}$.

▶ **Exercise 1.2.** Show that if E is an elementary set then there exist $B_1, \dots, B_n$ *pairwise disjoint* boxes such that $E = B_1 \uplus \cdots \uplus B_n$.

It would be tempting to define the measure of $E = B_1 \uplus \cdots \uplus B_n$ as $m(B_1) + \cdots + m(B_n)$. But we are not guarantied that the decomposition is a unique one.

• **Proposition 1.2** (Discretisation Formula)**.** *Let I be an interval. Then,*

$$m(I) = \lim_{n \to \infty} \frac{1}{n} \#(I \cap \tfrac{1}{n}\mathbb{Z}).$$

Consequently, if $B \subset \mathbb{R}^d$ is a box then,

$$m(B) = \lim_{n \to \infty} \frac{1}{n^d} \#(B \cap \tfrac{1}{n}\mathbb{Z}^d),$$

and if $E = B_1 \uplus \cdots \uplus B_n \subset \mathbb{R}^d$ is an elementary set then

$$m(B_1) + \cdots + m(B_n) = \lim_{n \to \infty} \frac{1}{n^d} \#(E \cap \tfrac{1}{n}\mathbb{Z}^d).$$

Proof. Let $a < b$ and I be an interval with endpoints a, b. For n large enough, note that

$$\begin{aligned} \#(I \cap \tfrac{1}{n}\mathbb{Z}) &= \#\left\{z \in \mathbb{Z} \ : \ \tfrac{z}{n} \in I\right\} \leq \#\left\{z \in \mathbb{Z} \ : \ na \leq z \leq nb\right\} \\ &\leq \#\left\{z \in \mathbb{Z} \ : \ \lfloor na \rfloor \leq z \leq \lceil nb \rceil\right\} \leq nb + 1 - (na - 1) + 1 = n(b - a) + 3. \end{aligned}$$

Similarly,

$$\#(I \cap \tfrac{1}{n}\mathbb{Z}) \geq \#\left\{z \in \mathbb{Z} \ : \ na < z < nb\right\} \geq n(b - a) - 1.$$

Dividing by n and taking $n \to \infty$ we obtain the formula for intervals.

Now, if $B = I_1 \times \cdots \times I_d$ then

$$B \cap \tfrac{1}{n}\mathbb{Z}^d = (I_1 \cap \tfrac{1}{n}\mathbb{Z}) \times \cdots \times (I_d \cap \tfrac{1}{n}\mathbb{Z}),$$

which implies that

$$\#(B \cap \tfrac{1}{n}\mathbb{Z}^d) = \prod_{j=1}^{d} \#(I_j \cap \tfrac{1}{n}\mathbb{Z}).$$

Dividing by n^d and taking $n \to \infty$ gives the formula for boxes.

The final assertion is a consequence of the fact that if $A \cap B = \emptyset$ then

$$(A \uplus B) \cap \tfrac{1}{n}\mathbb{Z}^d = (A \cap \tfrac{1}{n}\mathbb{Z}^d) \uplus (B \cap \tfrac{1}{n}\mathbb{Z}^d).$$

□

• **Definition 1.3.** If $E = B_1 \uplus \cdots \uplus B_n$ is an elementary set we define the measure of E to be

$$m(E) = m(B_1) + \cdots + m(B_n) = \lim_{n\to\infty} \frac{1}{n^d} \#(E \cap \tfrac{1}{n}\mathbb{Z}),$$

which is well defined by Proposition **??**.

► **Exercise 1.3.** Give an example of a set $A \subset [0,1]$ such that the limit

$$\lim_{n\to\infty} \frac{1}{n} \#(A \cap \tfrac{1}{n}\mathbb{Z})$$

does not exist.

$$\lim_{n\to\infty} \frac{1}{n} \#(A \cap \tfrac{1}{n}\mathbb{Z}) \qquad \text{and} \qquad \lim_{n\to\infty} \frac{1}{n} \#((A+x) \cap \tfrac{1}{n}\mathbb{Z})$$

both exist but are not equal.

♣ **Solution to ex:1.3.** :(

For the first example: Set

$$A = \left\{ \tfrac{2m-1}{2^n} \ : \ \mathbb{Z} \ni n, m \geq 1 \ , \ 2m-1 \leq 2^n \right\}.$$

Since for any $z \geq 1$ we can write $z = (2m-1)2^k$ for some $k \geq 0$ and $m \geq 1$, we have that

$$\#(A \cap 2^{-n}\mathbb{Z}) \geq \# \left\{ \tfrac{z}{2^n} \ : \ 1 \leq z \leq 2^n \right\} = 2^n.$$

So

$$\limsup_{n\to\infty} \tfrac{1}{n}\#(A\cap \tfrac{1}{n}\mathbb{Z}) \geq \limsup_{n\to\infty} 2^{-n}\#(A\cap 2^{-n}\mathbb{Z}) = 1.$$

On the other hand, if $x \in A \cap 3^{-k}\mathbb{Z}$ then for some $m, n \geq 1$ and $z \in \mathbb{Z}$, $x = \frac{2m-1}{2^n} = \frac{z}{3^k}$ which implies that $2^n z = 3^k(2m-1)$. The left hand side is even while the right hand side is odd, which is a contradiction. So $A \cap 3^{-k}\mathbb{Z} = \emptyset$ for all k, and we conclude that

$$\liminf_{n\to\infty} \tfrac{1}{n}\#(A\cap \tfrac{1}{n}\mathbb{Z}) \leq \liminf_{n\to\infty} 3^{-n}\#(A\cap 3^{-n}\mathbb{Z}) = 0.$$

For the second example take $A = \mathbb{Q} \cap [0,1]$ and $x = \pi$. :) ✓

This last exercise shows that we do not want to use the discretization formula to define the measure of general sets, but we can use it for more than just elementary sets.

► **Exercise 1.4.** Show that the measure of elementary sets has the following properties.

- $m : \mathcal{E}_0 \to [0,\infty)$.
- (Additivity) $m(E \uplus F) = m(E) + m(F)$.
- $m(\emptyset) = 0$.
- If B is a box, then $m(B) = |B|$.
- (Monotonicity) If $E \subset F$ then $m(E) \leq m(F)$.
- (Subadditivity) $m(E \cup F) \leq m(E) + m(F)$.
- (Translation invariance) For any $x \in \mathbb{R}^d$, $m(E + x) = m(E)$.

♣ **Solution to ex:1.4.** :(

The discretization formula guaranties that m is non-negative and additive, $m(\emptyset) = 0$ and also that if B is a box then $m(B) = |B|$.

For monotonicity recall that $F \setminus E$ is elementary, and $F = (F \cap E) \uplus (F \setminus E)$. Since $E \subset F$ we have that $F \cap E = E$. So $m(F) = m(E) + m(F \setminus E) \geq m(E)$.

Subadditivity follows from $E \cup F = E \uplus (F \setminus E)$ so

$$m(E \cup F) = m(E) + m(F \setminus E) \leq m(E) + m(F).$$

Translation invariance is another consequence of the discretization formula: For any interval I with endpoints $a < b$, we have that $\frac{z}{n} - x \in I$ if and only if $z \in nI + nx$. Thus, if $\frac{z}{n} - x \in I$ then $z \in [\lfloor na + nx \rfloor, \lceil nb + nx \rceil] \cap \mathbb{Z}$ and if $z \in [\lceil na + nx \rceil, \lfloor nb + nx \rfloor] \cap \mathbb{Z}$ then $\frac{z}{n} - x \in I$. Since

$$\#([\lfloor na+nx \rfloor, \lceil nb+nx \rceil] \cap \mathbb{Z}) \leq n(b-a)+3 \qquad \text{and} \qquad \#([\lceil na+nx \rceil, \lfloor nb+nx \rfloor] \cap \mathbb{Z}) \geq n(b-a)-1,$$

we have

$$\left| \# \left\{ z \in \mathbb{Z} \ : \ \tfrac{z}{n} - x \in I \right\} - n(b-a) \right| \leq 3.$$

Dividing by n and taking a limit we get that

$$m(I + x) = \lim_{n \to \infty} \tfrac{1}{n} \# \left\{ z \in \mathbb{Z} \ : \ \tfrac{z}{n} - x \in I \right\} = b - a = m(I).$$

If $B = I_1 \times \cdots \times I_d$ is a box in $\mathbb{R}^d$, then for any $x \in \mathbb{R}^d$, $B + x = (I_1 + x_1) \times \cdots \times (I_d + x_d)$, so $m(B + x) = \prod_{j=1}^d m(I_j + x_j) = \prod_{j=1}^d m(I) = m(B)$.

Finally, if $E = B_1 \uplus \cdots \uplus B_k$ is an elementary set, then $E + x = (B_1 + x) \uplus \cdots \uplus (B_k + x)$. So $m(E + x) = m(E)$. :) ✓

► **Exercise 1.5.** Show that if $m' : \mathcal{E}_0 \to [0, \infty)$ is additive and translation invariant, *i.e.* $m(E \uplus F) = m(E) + m(F)$ and $m(E + x) = m(E)$, then there exists a constant $c \geq 0$ such that $m' = cm$.

♣ **Solution to ex:1.5.** :(

If $E = B_1 \uplus \cdots \uplus B_k$ is an elementary set then $m'(E) = m'(B_1) + \cdots + m'(B_k)$. So we only need to show that $m'(B) = c|B|$ for any box B.

First, note that since

$$[0,1) = \biguplus_{j=1}^n \tfrac{1}{n}[j-1, j) \qquad \text{and} \qquad [0, \tfrac{n}{k}) = \biguplus_{j=1}^n \tfrac{1}{k}[j-1, j),$$

additivity and translation invariance guaranty that

$$m'([0, \tfrac{1}{n})^d) = \tfrac{1}{n^d} m'([0,1)^d) \qquad \text{and} \qquad m'([0,q)^d) = q^d \cdot m'([0,1))$$

for any $q \in \mathbb{Q}$ (make sure to fill in the details here). By translation invariance, for any $a_j \le b_j \in \mathbb{Q}$ we then have

$$m'([a_1, b_1) \times \cdots \times [a_d, b_d)) = \prod_{j=1}^{d} (b_j - a_j) \cdot m'([0,1)^d).$$

Set $C = m'([0,1)^d)$.

Let $B = I_1 \times \cdots \times I_d$. Assume that the endpoints of I_j are $a_j < b_j$.

Fix $\varepsilon > 0$ such that for all j we have $b_j - a_j > 2\varepsilon$. Let $a_j^-, a_j^+, b_j^-, b_j^+$ be *rational* numbers such that $a_j^+ \in (a_j, a_j + \varepsilon)$, $a_j^- \in (a_j - \varepsilon, a_j)$, $b_j^+ \in (b_j, b_j + \varepsilon)$, $b_j^- \in (b_j - \varepsilon, b_j)$. Let $I_j^+ = [a_j^-, b_j^+)$ and $I_j^- = [a_j^+, b_j^-)$. Let $B^\pm = I_1^\pm \times \cdots \times I_d^\pm$, so $B^- \subset B \subset B^+$. Since m' is additive and non-negative, it is also monotone. Monotonicity tells us that $m'(B^-) \le m'(B) \le m'(B^+)$.

Because $a_j^\pm, b_j^\pm \in \mathbb{Q}$ we get that

$$m'(B^+) = \prod_{j=1}^{d} (b_j^+ - a_j^-) \cdot C \le \prod_{j=1}^{d} (b_j - a_j + 2\varepsilon) \cdot C,$$

$$m'(B^-) = \prod_{j=1}^{d} (b_j^- - a_j^+) \cdot C \ge \prod_{j=1}^{d} (b_j - a_j - 2\varepsilon) \cdot C.$$

Setting $M = \max_j (b_j - a_j)$ we have that

$$m'(B^+) - \prod_{j-1}^{d} (b_j - a_j) \cdot C \le C \cdot 2\varepsilon \cdot dM^{d-1},$$

$$\prod_{j-1}^{d} (b_j - a_j) \cdot C - m'(B^-) \le C \cdot 2\varepsilon \cdot dM^{d-1}.$$

Taking $\varepsilon \to 0$ we get that

$$m'(B) = \prod_{j=1}^{d} (b_j - a_j) \cdot C = m'([0,1)^d) \cdot m(B).$$

:) ✓

► **Exercise 1.6.** Let $E \subset \mathbb{R}^d$ and $F \subset \mathbb{R}^k$ be elementary sets. Show that $E \times F$ is elementary (in $\mathbb{R}^{d+k}$) and that $m(E \times F) = m(E) \cdot m(F)$.

Number of exercises in lecture: 6

Total number of exercises until here: 6

Measure Theory

Ariel Yadin

Lecture 2: Jordan measure

2.1. Jordan measure

We have seen that the measure of elementary sets is a good way to measure length, area and volume for squares and rectangles, and anything that can be composed of finite unions of such. How about measuring a triangle? Or a circle? We saw that the discretisation formula also has its limitations. However, *Jordan measure* is essentially the way to use the discretisation formula.

Jordan measure is essentially approximating general shapes by elementary ones.

• **Definition 2.1** (Jordan measure)**.** Let $A \subset \mathbb{R}^d$ be a bounded set (*i.e.* $A \subset B(0,r) := \{x \ : \ |x| \leq r\}$). Define:

- The **Jordan inner measure** of A

$$J_*(A) := \sup_{\mathcal{E}_0 \ni E \subset A} m(A).$$

- The **Jordan outer measure** of A

$$J^*(A) := \inf_{\mathcal{E}_0 \ni F \supset A} m(F).$$

- If A is such that $J_*(A) = J^*(A)$ we say that A is **Jordan measurable**. We then define the **Jordan measure** of A as this common value, $m(A) = J_*(A) = J^*(A)$.

► **Exercise 2.1.** Show that J^*, J_* are monotone. Show that J^* is subadditive.

♣ **Solution to ex:2.1.** :(

Let $A \subset C$. Let $(E_n)_n, (F_n)_n$ be sequences of elementary sets such that $E_n \subset A, C \subset F_n$

and $m(E_n) \to J_*(A), m(F_n) \to J^*(C)$. Note that for every n,

$$J_*(C) \geq m(E_n) \to J_*(A) \qquad \text{and} \qquad J^*(A) \leq m(F_n) \to J^*(C).$$

Also, if $(G_n)_n$ is a sequence of elementary sets such that $A \subset G_n$ and $m(G_n) \to J^*(A)$, then $A \cup C \subset G_n \cup F_n$ so

$$J^*(A \cup C) \leq m(G_n \cup F_n) \leq m(G_n) + m(F_n) \to J^*(A) + J^*(C).$$

:) ✓

Note that every elementary set E is Jordan measurable and $m(E) = J^*(E) = J_*(E)$. Jordan sets are those which are "almost elementary".

► **Exercise 2.2.** Show that the following are equivalent for bounded $A \subset \mathbb{R}^d$.

- A is Jordan measurable.
- For every $\varepsilon > 0$ there exist elementary sets $E \subset A \subset F$, $E, F \in \mathcal{E}_0$, such that $m(F \setminus E) < \varepsilon$.
- For every $\varepsilon > 0$ there exists an elementary set $\mathcal{E}_0 \ni E \subset A$ such that $J^*(A \setminus E) < \varepsilon$.

♣ **Solution to ex:2.2.** :(

For any $\varepsilon > 0$ there exist elementary sets $E \subset A \subset F$ such that $m(E) > J_*(A) - \varepsilon$ and $m(F) < J^*(A) + \varepsilon$. So if A is Jordan measurable then $m(F) < J^*(A) + \varepsilon = J_*(A) + \varepsilon < m(E) + 2\varepsilon$. Since $E \subset F$ we have that $m(F) = m(E) + m(F \setminus E)$, so $m(F \setminus E) < 2\varepsilon$.

Now assume that there exist $E, F \in \mathcal{E}_0$, $E \subset A \subset F$ such that $m(F \setminus E) < \varepsilon$. Then, $A \setminus E \subset F \setminus E$, so $J^*(A \setminus E) \leq m(F \setminus E) < \varepsilon$.

Finally assume that for every $\varepsilon > 0$ there exists an elementary set $E_\varepsilon \subset A$ such that $J^*(A \setminus E_\varepsilon) < \varepsilon$. Since $E_\varepsilon \subset A$ we have that $m(E_\varepsilon) \leq J_*(A)$. Using the subadditivity of J^* we get

$$J^*(A) \leq J^*(A \setminus E_\varepsilon) + m(E_\varepsilon) < \varepsilon + J_*(A).$$

Taking $\varepsilon \to 0$ and recalling that $J_*(A) \leq J^*(A)$ by definition, we have that $J^*(A) = J_*(A)$ and aso A is jordan measurable. :) ✓

► **Exercise 2.3.** Show that for Jordan measurable sets A, C the following holds.

- $A \cup C, A \cap C, A \setminus C, A \triangle C$ are all Jordan measurable.
- $m(A) \geq 0$.
- (Additivity) If $A \cap C = \emptyset$ then $m(A \uplus C) = m(A) + m(C)$.
- (Monotonicity) If $C \subset A$ then $m(C) \leq m(A)$.
- (Subadditivity) $m(A \cup C) \leq m(A) + m(C)$.
- (Translation invariance) $m(A + x) = m(A)$.

► **Exercise 2.4.** Show that a bounded set A is Jordan measurable if and only if for every $\varepsilon > 0$ there exists an elementary set E such that $A \subset E$ and $J^*(E \setminus A) < \varepsilon$.

♣ **Solution to ex:2.4.** :(

Let B be a box containing the bounded set A.

Suppose that there exists an elementary set $F \subset B \setminus A$ such that $J^*((B \setminus A) \setminus F) < \varepsilon$. Then, with $E = B \setminus F$ we have that $A \subset E$ and $J^*(E \setminus A) = J^*((B \setminus A) \setminus F) < \varepsilon$.

Thus we obtain that A is Jordan measurable iff $B \setminus A$ is Jordan measurable iff for every $\varepsilon > 0$ there exists an elementary set F, $F \subset B \setminus A$, such that $J^*((B \setminus A) \setminus F) < \varepsilon$ iff for every $\varepsilon > 0$ there exists an elementary set E, $A \subset E$, such that $J^*(E \setminus A) < \varepsilon$. :) ✓

► **Exercise 2.5.** [Tao, ex. 1.1.7] Let $B \subset \mathbb{R}^d$ be a closed box, and let $f : B \to \mathbb{R}$ be a continuous function. Show that

- The graph $\{(x, f(x)) \ : \ x \in B\}$ is Jordan measurable with Jordan measure 0.

- The volume under the graph $\{(x,t) \ : \ x \in B, 0 \le t \le f(x)\}$ is Jordan measurable.

♣ **Solution to ex:2.5.** :(

f is uniformly continuous in B; that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $||x-y|| < \delta$ we have $|f(x) - f(y)| < \varepsilon$.

Write $B = \bigcup_{n=1}^{N} B_\delta(x_n)$ where $(B_\delta(x_n))_n$ are disjoint boxes of diameter less than δ, with x_n the center of B_δ. Uniform continuity of f gives that for any $x \in B$ there exists (a unique) n such that $x \in B_\delta(x_n)$, and so $(x, f(x)) \in B_\delta(x_n) \times (f(x_n) - \varepsilon, f(x_n) + \varepsilon) =: Q_{\delta,n}$. Thus, $\{(x, f(x)) \ : \ x \in B\} \subset \bigcup Q_{\delta,n}$. Since $m(Q_{\delta,n}) = m(B_\delta(x_n)) \cdot 2\varepsilon$, we have that

$$J^*(\{(x, f(x)) \ : \ x \in B\}) \le \sum_n m(Q_{\delta,n}) \le 2\varepsilon \sum_n m(B_\delta(x_n)) = 2\varepsilon m(B).$$

Taking $\varepsilon \to 0$ gives the first assertion.

For the second assertion, let $V = \{(x,t) \ : \ x \in B, 0 \le t \le f(x)\}$. Fix $\varepsilon, \delta > 0$ and $B = \bigcup_n B_\delta(x_)$ as above. For every n set

$$m_n = \inf_{x \in B_\delta(x_n)} f(x) \qquad M_n = \sup_{x \in B_\delta(x_n)} f(x).$$

Note that $|M_n - m_n| \le \varepsilon$ by uniform continuity. Now, set $K_n = \lfloor \frac{m_n}{\varepsilon} \rfloor$. For $0 \le k \le K_n - 1$ set $Q_{n,k} = B_\delta(x_n) \times \varepsilon[k, k+1)$ and $\tilde{Q}_n = B_\delta(x_n) \times [\varepsilon K_n, M_n]$.

For any n, if $0 \le k < K_n$ and $(x,t) \in Q_{n,k}$ then $x \in B_\delta(x_n)$ and $t < \varepsilon K_n \le m_n \le f(x)$. So

$$\bigcup_n \bigcup_{k=0}^{K_n - 1} Q_{n,k} \subset V.$$

On the other hand, if $x \in B$ and $0 \le t \le f(x)$ then there exists n such that $x \in B_\delta(x_n)$ and so $f(x) \in [m_n, M_n]$. Thus, either $t < \varepsilon K_n$ in which case $(x,t) \in Q_{n,k}$ for some $0 \le k < K_n$ or $\varepsilon K_n \le t \le f(x) \le M_n$ in which case $(x,t) \in \tilde{Q}_n$. Thus,

$$\bigcup_n \bigcup_{k=0}^{K_n - 1} Q_{n,k} \subset V \subset \bigcup_n \bigcup_{k=0}^{K_n - 1} Q_{n,k} \biguplus \bigcup_n \tilde{Q}_n.$$

Now, let $\Lambda := m(\bigcup_n \bigcup_{k=0}^{K_n-1} Q_{n,k})$. Then,

$$\begin{aligned} J^*(V) &\leq \Lambda + \sum_n m(\tilde{Q}_n) = \Lambda + \sum_n m(B_\delta(x_n)) \cdot (M_n - \varepsilon K_n) \\ &\leq \Lambda + \sum_n m(B_\delta(x_n)) \cdot (M_n - m_n + \varepsilon) \leq \Lambda + m(B) \cdot 2\varepsilon \\ &\leq J_*(V) + m(B) \cdot 2\varepsilon. \end{aligned}$$

Taking $\varepsilon \to 0$ completes the proof. :) ✓

► **Exercise 2.6.** [Tao, p. 31, ex. 1.2.6] Show that it is not true that

$$J^*(E) = \sup_{E \supset U \text{ open}} J^*(U).$$

♣ **Solution to ex:2.6.** :(

Take $E = [0,1] \setminus \mathbb{Q}$. If $U \subset E$ is an open set, then $U = \emptyset$, because E does not contain any open intervals. Thus, $\sup_{U \subset E, U \text{ open}} m^*(U) = 0$. However, if $E \subset \bigcup_n I_n$ where $(I_n)_n$ are disjoint intervals, then it must be that $[0,1] \subset \bigcup_n I_n$. So $\sum_n m(I_n) \geq m([0,1]) = 1$. Thus, $m^*(E) = 1$. :) ✓

► **Exercise 2.7.** [Tao, p. 12, ex. 1.1.13] Prove the discretisation formula

$$m(A) = \lim_{N \to \infty} \tfrac{1}{N^d} \# \left(A \cap \tfrac{1}{N} \mathbb{Z}^d \right)$$

for all Jordan measurable subsets $A \subset \mathbb{R}^d$.

♣ **Solution to ex:2.7.** :(

We know the discretisation formula holds for all elementary sets. For simplicity of the presentation, let us denote $I_N(A) = \frac{1}{N^d} \# \left(A \cap \frac{1}{N} \mathbb{Z}^d \right)$.

Let A be Jordan measurable, and let $\varepsilon > 0$. Let $E \subset A \subset F$ be elementary sets such that $m(F) < m(A) + \varepsilon$ and $m(E) > m(A) - \varepsilon$. Note that

$$E \cap \tfrac{1}{N}\mathbb{Z}^d \subset A \cap \tfrac{1}{N}\mathbb{Z}^d \subset F \cap \tfrac{1}{N}\mathbb{Z}^d,$$

so $I_N(E) \leq I_N(A) \leq I_N(F)$. Taking $N \to \infty$,

$$\begin{aligned} m(A) < m(E) + \varepsilon = \varepsilon + \lim_{N\to\infty} I_N(E) \leq \varepsilon + \lim_{N\to\infty} I_N(A) \\ \leq \varepsilon + \lim_{N\to\infty} I_N(F) = \varepsilon + m(F) < 2\varepsilon + m(A). \end{aligned}$$

Taking $\varepsilon \to 0$ we get the formula. :) ✓

► **Exercise 2.8.** [Tao, p. 12, ex. 1.1.14] A *dyadic cube of scale* 2^{-n} is a box of the form:

$$D_n(z_1, \ldots, z_d) := \left[\tfrac{z_1}{2^n}, \tfrac{z_1+1}{2^n}\right) \times \cdots \times \left[\tfrac{z_d}{2^n}, \tfrac{z_d+1}{2^n}\right),$$

For $n \in \mathbb{N}$ and $(z_1, \ldots, z_d) \in \mathbb{Z}^d$. Note that these are half-open half-closed.

Let $\mathcal{E}_*(A, n)$ be the number of dyadic cubes of scale 2^{-n} that are *contained* in A, and let $\mathcal{E}^*(A, n)$ be the number of dyadic cubes of scale 2^{-n} that *intersect* A.

Show that a bounded set A is Jordan measurable if and only if

$$\lim_{n\to\infty} 2^{-dn}(\mathcal{E}^*(A,n) - \mathcal{E}_*(A,n)) = 0.$$

Show that Jordan measurable sets admit

$$m(A) = \lim_{n\to\infty} 2^{-dn}\mathcal{E}^*(A,n) = \lim_{n\to\infty} 2^{-dn}\mathcal{E}_*(A,n).$$

♣ **Solution to ex:2.8.** :(

If $\lim_{n\to\infty} 2^{-dn}(\mathcal{E}^*(A,n) - \mathcal{E}_*(A,n)) = 0$: Let $S_*(A,n)$ be the set of dyadic cubes of scale 2^{-n} that are *contained* in A, and let $S^*(A,n)$ be the set of dyadic cubes of scale 2^{-n} that *intersect* A. Then,

$$\bigcup S_*(A,n) \subset A \subset \bigcup S^*(A,n),$$

and these are elementary sets. Since dyadic cubes of the same scale are always disjoint, we get that

$$2^{-dn}\mathcal{E}_*(A,n) = m(\bigcup S_*(A,n)) \leq J_*(A) \leq J^*(A) \leq m(\bigcup S^*(A,n)) = 2^{-dn}\mathcal{E}^*(A,n).$$

Thus, taking $n \to \infty$ we get that $J^*(A) = J_*(A)$. Moreover, we have that

$$m(A) = \lim_{n\to\infty} 2^{-dn}\mathcal{E}^*(A,n) = \lim_{n\to\infty} 2^{-dn}\mathcal{E}_*(A,n).$$

If A is Jordan measurable: Let $B = I_1 \times \cdots \times I_d$ be a box where I_j is the interval with endpoints $a_j < b_j$, define

- $(B)^n := [a_1 - 2^{-n}, b_1 + 2^{-n}] \times \cdots \times [a_d - 2^{-n}, b_d + 2^{-n}]$, the enlargement of B by 2^{-n} in each coordinate direction.
- $(B)_n := (a_1 + 2^{-n}, b_1 - 2^{-n}) \times \cdots \times (a_d + 2^{-n}, b_d - 2^{-n})$, shrinking B by 2^{-n} in each coordinate direction.

Under these definitions, if a dyadic cube $D_n(z)$ of scale 2^{-n} intersects B then it is contained in $(B)^n$; if $D_n(z)$ is not contained in B then $D_n(z)$ does not intersect $(B)_n$.

Finally note that

$$\begin{aligned} m((B)^n) - m((B)_n) &= \prod_{j=1}^{d}(b_j - a_j + 2\cdot 2^{-n}) - \prod_{j=1}^{d}(b_j - a_j - 2\cdot 2^{-n}) \\ &= \sum_{S\subset\{1,\ldots,d\}} \prod_{j\in S}(b_j - a_j)\cdot\Big((2\cdot 2^{-n})^{d-|S|} - (-2\cdot 2^{-n})^{d-|S|}\Big) \\ &\leq 2^d \cdot 4 \cdot 2^{-n} \cdot m(B). \end{aligned}$$

Fix $\varepsilon > 0$ and let $E \subset A \subset F$ be elementary sets such that $m(F) \leq m(A) + \varepsilon$ and $m(E) \geq m(A) - \varepsilon$. Suppose that $E = \biguplus_{k=1}^{n} E_k$ and $F = \biguplus_{j=1}^{m} F_j$ where E_n, F_j are boxes.

Let $D_n(z)$ be a dyadic cube of scale 2^{-n} such that $D_n(z)$ intersects A but is not contained in A. Then, there exists j such that $D_n(z)$ intersects some F_j, so $D_n(z)$ is contained in $(F_j)^n$. Also, $D_n(z)$ is not contained in E_k for all k, so for all k we get that $D_n(z)$ does not intersect $(E_k)_n$. Thus, if we define

$$(E)_n := \biguplus_{k=1}^{n}(E_k)_n \qquad \text{and} \qquad (F)^n := \bigcup_{j=1}^{m}(F_j)^n$$

then

$$m((E)_n) = \sum_{k=1}^{n} m((E_k)_n) \geq \sum_{k=1}^{n} m(E_k) \cdot \left(1 - 2^d \cdot 2^{2-n}\right) = m(E) \cdot \left(1 - 2^d \cdot 2^{2-n}\right),$$

$$m((F)^n) \leq \sum_{j=1}^{m} m((F_j)^n) \leq \sum_{j=1}^{m} m(F_j) \cdot \left(1 + 2^d \cdot 2^{2-n}\right) = m(F) \cdot \left(1 + 2^d \cdot 2^{2-n}\right),$$

and since for any Jordan measurable set J we have that $2^{-dn}\mathcal{E}_*(J) \leq m(J) \leq 2^{-dn}\mathcal{E}^*(J)$,

$$\begin{aligned} 2^{-dn}(\mathcal{E}^*(A,n) - \mathcal{E}_*(A,n)) \leq 2^{-dn}(\mathcal{E}_*((F)^n, n) - \mathcal{E}^*((E)_n, n)) &\leq m((F)^n) - m((E)_n) \\ &\leq m(F) - m(E) + 2^d \cdot 2^{2-n} \cdot (m(F) + m(E)) \\ &\leq \varepsilon + 2^d \cdot 2^{2-n} \cdot (2m(A) + \varepsilon). \end{aligned}$$

Taking $n \to \infty$, we have that for all $\varepsilon > 0$,

$$0 \leq \lim_{n\to\infty} 2^{-dn}(\mathcal{E}^*(A,n) - \mathcal{E}_*(A,n)) \leq \varepsilon,$$

so this limit must be 0. :) ✓

► **Exercise 2.9.** [Tao, p. 13, ex. 1.1.18] Show that for any bounded set A:

- $J^*(\bar{A}) = J^*(A)$.
- $J_*(A^\circ) = J_*(A)$.
- A is Jordan measurable if and only if $J^*(\partial A) = 0$.

♣ **Solution to ex:2.9.** :(

First, $A \subset \bar{A}$ so $J^*(A) \leq J^*(\bar{A})$, and we only need to prove $J^*(\bar{A}) \leq J^*(A)$. Let $\varepsilon > 0$. Then choose disjoint boxes $(B_n)_{n=1}^N$ such that $A \subset \bigcup_n B_n$ and $\sum_n m(B_n) \leq J^*(A) + \varepsilon$. Note that

$$\bar{A} \subset \overline{\bigcup_n B_n} \subset \bigcup_n \bar{B}_n,$$

and since $\bar{B}_n$ are also boxes,

$$J^*(\bar{A}) \leq \sum_n m(\bar{B}_n) = \sum_n m(B_n) \leq J^*(A) + \varepsilon.$$

Taking $\varepsilon \to 0$ completes the proof of the first item.

For the second item, since $A^\circ \subset A$ we only need to show that $J_*(A) \leq J_*(A^\circ)$. Fix $\varepsilon > 0$ and let $(B_n)_{n=1}^N$ be a finite number of disjoint boxes such that $\bigcup_n B_n \subset A$ and $\sum_n m(B_n) \geq J_*(A) - \varepsilon$. Then, since B_n° are disjoint boxes, and since $m(B_n^\circ) = m(B_n)$, from $\bigcup_n B_n^\circ \subset A^\circ$ we deduce that

$$J_*(A^\circ) \geq \sum_n m(B_n^\circ) = \sum_n m(B_n) \geq J_*(A) - \varepsilon.$$

Taking $\varepsilon \to 0$ completes the second item.

For the final item: First, note that $(\partial A)^\circ = \emptyset$. If $D_n(z) \subset \partial A$ for some dyadic cube $D_n(z)$, then $(\partial A)^\circ \supset (D_n(z))^\circ \neq \emptyset$, a contradiction. So ∂A contains no dyadic cubes, which is to say that $\mathcal{E}_*(\partial A, n) = 0$ for all n. Thus, ∂A is Jordan measurable with $J(\partial A) = 0$ if and only if $2^{-dn}\mathcal{E}^*(\partial A, n) \to 0$.

Note that if $D_n(z)$ is a dyadic cube, then it intersects A and is not contained in A, if and only if it intersects ∂A. That is, $\mathcal{E}^*(\partial A, n) = \mathcal{E}^*(A, n) - \mathcal{E}_*(A, n)$.

Now, by a previous exercise, A is Jordan measurable, if and only if $2^{-dn}(\mathcal{E}^*(A, n) - \mathcal{E}_*(A, n)) \to 0$ if and only if $2^{-dn}\mathcal{E}^*(\partial A, n) \to 0$, if and only if ∂A is Jordan measurable with $J(\partial A) = 0$, if and only if $J^*(\partial A) = 0$.

The last step follows from the fact that if $J^*(\partial A) = 0$ then $J_*(\partial A) \leq J^*(\partial A) = 0$ and so ∂A is Jordan measurable with $J(\partial A) = 0$. :) ✓

► **Exercise 2.10.** Show that any triangle ABC in $\mathbb{R}^2$ is Jordan measurable.

Show that any compact convex polygon in $\mathbb{R}^2$ is Jordan measurable.

► **Exercise 2.11.** Show that the ball $B(x, r) = \{x \in \mathbb{R}^d \ : \ |x| < r\}$ is Jordan measurable with measure $m(B(x, r)) = c_d r^d$ where $c_d > 0$ is a constant depending only on the dimension d.

▶ **Exercise 2.12.** Show that if $A \subset \mathbb{R}^d, C \subset \mathbb{R}^k$ are Jordan measurable, then $A \times C$ is Jordan measurable and $m(A \times C) = m(A) \cdot m(C)$.

♣ **Solution to ex:2.12.** :(

Fix $\varepsilon > 0$. Let $E \subset A \subset F, E' \subset C \subset F'$ such that $E, E', F, F' \in \mathcal{E}_0$ and $m(F \setminus E) < \varepsilon$, $m(F' \setminus E') < \varepsilon$. Note that $E \times E' \subset A \times C \subset F \times F'$ and $E \times E', F \times F'$ are elementary sets. Thus, since we assumed that A, C are Jordan measurable,

$$J^*(A \times C) \leq m(F \times F') = m(F) \cdot m(F') < (m(E) + \varepsilon) \cdot (m(E') + \varepsilon) \leq (m(A) + \varepsilon) \cdot (m(C) + \varepsilon).$$

Also, since $m(A) = m(A \setminus E) + m(E) \leq m(F \setminus E) + m(E)$ and $m(C) = m(C \setminus E') + m(E') \leq m(F' \setminus E') + m(E')$,

$$J_*(A \times C) \geq m(E \times E') \geq (m(A) - \varepsilon) \cdot (m(C) - \varepsilon).$$

Taking $\varepsilon \to 0$ we have that $J^*(A \times C) = J_*(A \times C) = m(A) \cdot m(C)$. :) ✓

▶ **Exercise 2.13.** Show that if $J^*(A) = 0$ then A is Jordan measurable.

Show that if $m(A) = 0$ for Jordan measurable A, then any $C \subset A$ is Jordan measurable.

▶ **Exercise 2.14.** give an example of a sequence $(A_n)_n$ of Jordan measurable sets such that $A := \bigcup_n A_n$ is bounded but *not* Jordan measurable.

●●● **Theorem 2.2.** *Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be an invertible linear transformation. If A is Jordan measurable, then $L(A)$ is also, and $m(L(A)) = |\det L| m(A)$.*

Proof. If E is an elementary set, then $L(E)$ is Jordan measurable. Indeed, recall that any invertible matrix L can be written as a product of *elementary operation matrices*: $L = T_1 \cdots T_n$ where each T_j is either multiplication of a row by a scalar c, addition of one row to another row, or swapping of two rows. That is, T_j is in one of the following families of matrices:

$$M_{c,j} = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ \vdots & \ddots & & & \vdots \\ & \cdots & c & \cdots & \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & & 1 \end{bmatrix} \qquad R_{i,j} = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ \vdots & \ddots & & & \vdots \\ \cdots 1 & 0 \cdots & 1 & \cdots & \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & & 1 \end{bmatrix} \qquad S_{i,j} = \begin{bmatrix} 1 & 0 & \cdots & & & 0 \\ \vdots & \ddots & & & & \vdots \\ & \cdots & 0 & 1 & \cdots & \\ & \cdots & 1 & 0 & \cdots & \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & & & & 1 \end{bmatrix}$$

That is $M_{c,j}$ is obtain by multiplying the j-th row of the identity matrix by c; $R_{i,j}$ is obtained by adding the i-th row in the identity matrix to the j-th row; $S_{i,j}$ is obtained by swapping the i-th and j-th rows of the identity matrix.

It is immediate to compute that $\det M_{c,j} = c, |\det R_{i,j}| = |\det S_{i,j}| = 1$. Also, if $B = I_1 \times \cdots \times I_d$ we have that $M_{c,j}(B) = I_1 \times \cdots \times cI_j \times \cdots \times I_d$ and for $i < j$, $S_{i,j}(B) = I_1 \times \cdots I_j \times \cdots \times I_i \times \cdots \times I_d$. Thus, we have that for any box B, and any linear map $L \in \{M_{c,j}, S_{i,j}\}$, $m(L(B)) = |\det L| \cdot m(B)$. If $E = B_1 \uplus \cdots \uplus B_k$ is an elementary set, where $(B_j)_{j=1}^k$ are boxes, then $L(E) = L(B_1) \uplus \cdots \uplus L(B_k)$, so $L(E)$ is an elementary set and

$$m(L(E)) = \sum_{j=1}^k m(L(B_j)) = \sum_{j=1}^k |\det L| \cdot m(B_j) = |\det L| \cdot m(E).$$

Finally, if A is Jordan measurable, the for any $\varepsilon > 0$ choose elementary sets $E \subset A \subset F$ such that $m(F) < m(E) + \varepsilon$. So, $L(E) \subset L(A) \subset L(F)$ and

$$\begin{aligned} J^*(L(A)) &\leq m(L(F)) = |\det L| \cdot m(F) \leq |\det L| \cdot m(E) + |\det L| \cdot \varepsilon \\ &\leq |\det L| \cdot J(A) + |\det L| \cdot \varepsilon \leq |\det L| \cdot m(F) + |\det L| \cdot \varepsilon \\ &\leq |\det L| \cdot m(E) + |\det L| \cdot 2\varepsilon = m(L(E)) + |\det L| \cdot 2\varepsilon \leq J_*(L(A)) + |\det L| \cdot 2\varepsilon. \end{aligned}$$

Taking $\varepsilon \to 0$ we have that $J_*(L(A)) = J^*(L(A)) = |\det L| \cdot J(A)$. This proves the theorem for the case that $L \in \{M_{c,j}, S_{i,j}\}$.

Now for a somewhat more cumbersome computation:

$$R_{i,j}(B) = \Big\{(x_1, \ldots, x_i, \ldots, x_i + x_j, \ldots, x_d) \in \mathbb{R}^d \ : \ \forall\ k\ , x_k \in I_k\Big\},$$

which is the Cartesian product of the parallelepiped $\{(x, x+y) \ : \ x \in I_i, y \in I_j\}$ with a box in $\mathbb{R}^{d-2}$. Indeed, if we apply $S = S_{i,1}S_{j,2}$ to this set,

$$\begin{aligned} R_{i,j}(B) &= \Big\{(x_1, \ldots, x_i, \ldots, x_i + x_j, \ldots, x_d) \in \mathbb{R}^d \ : \ \forall\ k\ , x_k \in I_k\Big\} \\ &= S\Big\{(x_i, x_i + x_j, x_3, \ldots, x_{i-1}, x_1, x_{i+1}, \ldots, x_{j-1}, x_2, x_{j+1}, \ldots, x_d) \in \mathbb{R}^d \ : \ \forall\ k\ , x_k \in I_k\Big\} \\ &= S(\{(x, x+y) \ : \ x \in I_i, y \in I_j\} \times I_3 \times \cdots I_{i-1} \times I_1 \times I_{i+1} \times \cdots \times I_{j-1} \times I_j \times I_{j+1} \times \cdots \times I_d). \end{aligned}$$

Since S preserves Jordan measure (and measurability), it suffices to show the Jordan measurability of $P := \{(x, x+y) \ : \ x \in I_i, y \in I_j\}$ and compute its Jordan measure. By translating P we may assume without loss of generality that the endpoints of I_i are $0 < a$ and the endpoints of I_j are $0, b$. We may also write $P = T_1 \uplus B \uplus T_2$ where T_1, T_2 are right-angle triangles, with orthogonal sides parallel to the axes, and B is a box, as in Figure 1. In a previous exercise it was shown that triangles are Jordan measurable. A translation of T_2 by the vector $-(0, b)$ shows that $T_1 \uplus T_2$ has the Jordan measure of the box $I_i \times I_i$. Thus, P is Jordan measurable and has Jordan measure $J(P) = |I_i| \cdot |I_j|$, which consequently is $J(P) = m(I_i \times I_j)$.

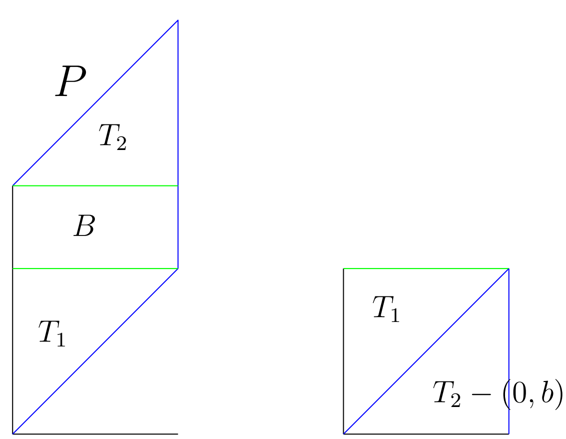

FIGURE 1. Computation of the Jordan measure of the parallelepiped P.

► **Exercise 2.15.** Let $P := \{(x, x+y) \ : \ x \in I, y \in J\}$ where I, J are intervals. Show that P is Jordan measurable and that $J(P) = |I| \cdot |J|$.

We conclude that $R_{i,j}(B)$ is Jordan measurable, and that

$$J(R_{i,j}(B)) = J(S(P \times B)) = J(P \times B) = J_2(P) \cdot J_{d-2}(B) = |I_i| \cdot |I_j| \cdot \prod_{k \neq i,j} |I_k| = m(B),$$

where $P = \{(x, x+y) \ : \ x \in I_i, y \in I_j\}$ and $B = I_3 \times \cdots I_{i-1} \times I_1 \times I_{i+1} \times \cdots \times I_{j-1} \times I_j \times I_{j+1} \times \cdots \times I_d \subset \mathbb{R}^{d-2}$. Since $|\det R_{i,j}| = 1$ we have just proved that for any box B, the set $R_{i,j}(B)$ is Jordan measurable and $J(R_{i,j}(B)) = |\det R_{i,j}| \cdot m(B)$.

Just as for $M_{c,j}$ and $S_{i,j}$ we now use the fact that $R_{i,j}(A \uplus B) = R_{i,j}(A) \uplus R_{i,j}(B)$ to prove the theorem for any $L \in \{M_{c,j}, S_{i,j}, R_{i,j}\}$.

Finally, if L is a general invertible linear map, then $L = L_1 \cdots L_n$ where $L_k \in \{M_{c,j}, S_{i,j}, R_{i,j}\}$ for all k. So for any A that is Jordan measurable, also $L_n(A)$ is Jordan measurable, and thus $L_{n-1}L_n(A)$ is as well, and so on to get that $L(A) = L_1 \cdots L_n(A)$ is Jordan measurable. We also compute the measure by

$$J(L(A)) = J(L_1(L_2 \cdots L_n(A))) = |\det L_1| \cdot J(L_2 \cdots L_n(A)) =$$
$$= \cdots = |\det L_1| \cdots |\det L_n| \cdot J(A) = |\det L| \cdot J(A).$$

□

Number of exercises in lecture: 15

Total number of exercises until here: 21

Measure Theory

Ariel Yadin

Lecture 3: Lebesgue outer measure

3.1. From finite to countable

Recall that in order to measure sets from outside we used the outer measure by approximating with finitely many elementary sets.

▶ **Exercise 3.1.** Show that

$$J^*(A) = \inf_{A \subset B_1 \cup \cdots \cup B_n} m(B_1) + \cdots + m(B_n),$$

where $B_1, \ldots, B_n$ are always boxes.

Lebesgue's idea is that there is no reason to stop with finite, and not *countable* collections. (Mathematicians are not afraid of infinity anymore...)

• **Definition 3.1.** Let $A \subset \mathbb{R}^d$ be a set. The **Lebesgue outer measure** of A is defined to be

$$m^*(A) := \inf \Big\{ \sum_n |B_n| \ : \ (B_n)_n \text{ is a sequence of boxes} \ , \ A \subset \bigcup_n B_n \Big\}.$$

✓ Note that we may have $m^*(A) = \infty$.

▶ **Exercise 3.2.** Show that in general $m^*(A) \leq J^*(A)$ for bounded sets A.
Show that for $Q := \mathbb{Q} \cap [0,1]$ we have that

$$J^*(Q) = 1 \qquad \text{and} \qquad m^*(Q) = 0.$$

♣ **Solution to ex:3.2.** :(

If A is bounded then the infimum for Lebesgue outer measure is on a larger set than the infimum for Jordan outer measure.

For Q, we have that $\overline{Q} = [0,1]$, so $J^*(\overline{Q}) = J^*([0,1]) = m([0,1]) = 1$ and if $Q = \{q_1, q_2, \ldots\}$ is an enumeration of Q then $m^*(Q) \leq \sum_n |\{q_n\}| = 0$. :) ✓

► **Exercise 3.3.** Give an example of an *unbounded* set with Lebesgue outer measure 0.

• **Proposition 3.2.** *Properties of Lebesgue outer measure:*

- $m^*(\emptyset) = 0$.
- *(Monotonicity) If* $A \subset C$ *then* $m^*(A) \leq m^*(C)$.
- *(Countable subadditivity) If* $(A_n)_n$ *is a sequence of sets then* $m^*(\bigcup_n A_n) \leq \sum_n m^*(A_n)$.

Proof. First, $\emptyset$ is a box of volume 0; another proof is by noting that $\emptyset \subset [0, \frac{1}{n}]^d$ for all n, so $m^*(\emptyset) \leq \inf_n m([0, \frac{1}{n}]^d) = 0$.

For $A \subset C$ we have that the infimum used to obtain $m^*(A)$ is over a larger set that the one used to obtain the infimum for $m^*(C)$.

Let $(A_n)_n$ be a sequence of sets. Fix $\varepsilon > 0$. For every n let $(B_{n,k})_k$ be a sequence of boxes such that $A_n \subset \bigcup_k B_{n,k}$ and

$$\sum_k |B_{n,k}| \leq m^*(A_n) + \varepsilon \cdot 2^{-n}.$$

Consider the sequence of boxes $(B_{n,k})_{n,k}$. We have that

$$\bigcup_n A_n \subset \bigcup_{n,k} B_{n,k},$$

and so

$$m^*(\bigcup_n A_n) \le \sum_{n,k} |B_{n,k}| \le \sum_n m^*(A_n) + \sum_n \varepsilon \cdot 2^{-n} = \sum_n m^*(A_n) + 2\varepsilon.$$

Taking $\varepsilon \to 0$ we have countable subadditivity. □

It is now natural to ask for the additivity property: if $A \cap C = \emptyset$ is it true that $m^*(A \uplus C) = m^*(A) + m^*(C)$? As it turns out, this is not always the case, a counter example will be given in the future. However, in some cases, when the sets are separated enough, we have additivity.

• **Proposition 3.3.** *If* A, C *are such that* $\mathrm{dist}(A, C) > 0$ *then* $m^*(A \uplus C) = m^*(A) + m^*(C)$.

Specifically, if A, C *are closed disjoint sets and* A *is compact then* $\mathrm{dist}(A, C) > 0$.

Proof. By sub-additivity it suffices to prove $m^*(A) + m^*(C) \le m^*(A \uplus C)$. We may also assume w.l.o.g. that $m^*(A \uplus C) < \infty$ and by monotonicity that $m^*(A) < \infty, m^*(C) < \infty$.

Fix $\varepsilon > 0$. Let $(B_n)_n$ be a sequence of boxes such that $A \uplus C \subset \bigcup_n B_n$ and $\sum_n |B_n| \le m^*(A \uplus C) + \varepsilon$.

Let $r = \mathrm{dist}(A, C) > 0$. Note that for any n, we may replace the box B_n by a finite number of disjoint boxes $B_{n,1}, \ldots, B_{n,k}$ such that the diameter of any $B_{n,j}$ is less than r. So we obtain a sequence $(B'_n)_n$ such that $A \uplus C \subset \bigcup_n B'_n$ and $\sum_n |B'_n| \le m^*(A \uplus C) + \varepsilon$, and such that any box B'_n cannot intersect both A and C.

Thus, we have that $\mathbb{N} = N_A \uplus N_C$ where $N_A = \{n : B'_n \cap A \neq \emptyset\}$ and $N_C = \{n : B'_n \cap C \neq \emptyset\}$.

We now have that

$$A \subset \bigcup_{n \in N_A} B'_n \qquad \text{and} \qquad C \subset \bigcup_{n \in N_C} B'_n,$$

and so

$$m^*(A) + m^*(C) \le \sum_{n \in N_A} |B'_n| + \sum_{n \in N_C} |B'_n| \le \sum_n |B'_n| \le m^*(A \uplus C) + \varepsilon.$$

Taking $\varepsilon \to 0$ completes the proof.

As for the case where A, C are closed and A is compact, note that $\mathrm{dist}(x, C)$ is a continuous function of x (because C is closed) and so achieves a minimum on A when A is compact. □

► **Exercise 3.4.** Give an example of two closed sets A, C such that $A \cap C = \emptyset$ but $\mathrm{dist}(A, C) = 0$.

The next proposition shows that the notation "m" is an appropriate one, as an extension of elementary measure.

• **Proposition 3.4.** *For any elementary set E we have that $m^*(E) = m(E)$.*

Proof. If suffices to prove that $J^*(E) \leq m^*(E)$.

Case I. E is closed. Since E is bounded, it is compact, and the Heine-Borel Theorem tells us that every open cover of E has a finite sub-cover.

Fix any $\varepsilon > 0$. Let $(B_n)_n$ be a sequence of boxes such that $E \subset \bigcup_n B_n$ and $\sum_n |B_n| \leq m^*(E) + \varepsilon$.

✓ The boxes $(B_n)_n$ do not form an open cover – they need not be open.

For every n let B'_n be an *open* box containing $B_n \subset B'_n$ such that $|B'_n| \leq |B_n| + \varepsilon 2^{-n}$. (Exercise: show this is possible.) Then, $\sum_n |B'_n| \leq m^*(E) + 2\varepsilon$ and $(B'_n)_n$ are an open cover of E. Thus, there is a finite sub-cover

$$E \subset \bigcup_{n=1}^{N} B'_n,$$

for some large enough N. Thus,

$$J^*(E) \leq \sum_{n=1}^{N} |B'_n| \leq \sum_n |B'_n| \leq m^*(E) + 2\varepsilon.$$

Taking $\varepsilon \to 0$ completes the proof for closed E.

Case II. E is a general elementary set. Write $E = B_1 \uplus \cdots \uplus B_n$ where $(B_j)_{j=1}^n$ are *disjoint* boxes. These need not be closed boxes.

Fix $\varepsilon > 0$. For every $1 \le j \le n$ let $B'_j \subset B_j$ be a *closed* box such that $|B'_j| \ge |B_j| - \frac{\varepsilon}{n}$. (Exercise: Show this is always possible.) Then, $E' = B'_1 \uplus \cdots \uplus B'_n$ is a finite union of disjoint closed boxes, and thus a closed elementary set. Also,

$$m(E') = \sum_{j=1}^{n} |B'_j| \ge \sum_{j=1}^{n} |B_j| - \varepsilon = m(E) - \varepsilon.$$

So by Case I,

$$m(E) \le m(E') + \varepsilon = m^*(E') + \varepsilon \le m^*(E) + \varepsilon.$$

Taking $\varepsilon \to 0$ completes the proof. □

► **Exercise 3.5.** Show that for any bounded set A we have

$$J_*(A) \le m^*(A) \le J^*(A).$$

Show that if A is Jordan measurable then $m(A) = m^*(A)$.

♣ **Solution to ex:3.5.** :(

We have already seen $m^*(A) \le J^*(A)$. If E is an elementary set such that $E \subset A$, then $m(E) = m^*(E) \le m^*(A)$ by monotonicity. Taking supremum over all such elementary sets contained in A, we get that $J_*(A) \le m^*(A)$. This proves the first assertion.

If A is Jordan measurable, then $m(A) = J_*(A) \le m^*(A) \le J^*(A) = m(A)$. :) ✓

Number of exercises in lecture: 5

Total number of exercises until here: 26

Measure Theory

Ariel Yadin

Lecture 4: Lebesgue measure

4.1. Definition of Lebesgue measure

Recall that A is Jordan measurable if and only if for every $\varepsilon > 0$ there exists an elementary set E such that $A \subset E$ and $J^*(E \setminus A) < \varepsilon$. This motivates the following. Later we will see that there is another way to approach the issue of Lebesgue measure, and outer measures in general.

• **Definition 4.1** (Lebesgue measure)**.** A set $A \subset \mathbb{R}^d$ is said to be **Lebesgue measurable** if for every $\varepsilon > 0$ there exists an *open* set U such that $A \subset U$ and $m^*(U \setminus A) < \varepsilon$.

for Lebesgue measurable A we denote $m(A) := m^*(A)$ and refer to this as the **Lebesgue measure** of A.

✓ Lebesgue measurable sets are sets that are "almost open".

? Why do *open* sets enter the picture?

• **Definition 4.2.** We say that boxes $(B_n)_n$ are **almost disjoint** if any two boxes can only intersect at their boundary; that is, for every $n \neq m$, $B_n^\circ \cap B_m^\circ = \emptyset$.

• **Lemma 4.3.** *Any open set $U \subset \mathbb{R}^d$ can be written as a countable union of almost disjoint closed boxes.*

Proof. For any $z \in \mathbb{Z}^d$ define the dyadic cube at scale $n \geq 0$:

$$D_n(z) = \left[\tfrac{z_1}{2^n}, \tfrac{z_1+1}{2^n}\right] \times \cdots \times \left[\tfrac{z_d}{2^n}, \tfrac{z_d+1}{2^n}\right].$$

This is the d-dimensional cube of side-lengths 2^{-n} and corner z.

The following is the crucial property to verify: If $D_n(z)^\circ \cap D_m(z')^\circ \neq \emptyset$ then one is contained in the other.

Specifically, if $D_n(z) \subsetneq D_m(z')$ then $n > m$.

Now, set

$$\Gamma = \left\{ (z,n) \ : \ z \in \mathbb{Z}^d, n \geq 0, \ \ \overline{D_n(z)} \subset U \right\}.$$

These are all dyadic cubes whose closure is contained in U. Also, set

$$\Lambda = \left\{ (z,n) \in \Gamma \ : \ \nexists \ (z',n') \in \Gamma \ , \ D_n(z) \subset D_{n'}(z') \right\}.$$

These are all dyadic cubes contained in U that are maximal with respect to inclusion.

We claim that

$$U = \bigcup_{(z,n)\in\Lambda} \overline{D_n(z)}.$$

One inclusion is obvious, since all closures of dyadic cubes indexed by Λ are contained in U. For the other inclusion, let $x \in U$. Then, since U is open (this is the only place we use that U is an open set!), there is a small ball $B(x,\varepsilon) \subset U$. However, for large enough n (so that $\sqrt{d}2^{-n} < \varepsilon$), if $x \in D_n(z)$ then $\overline{D_n(z)} \subset B(x,\varepsilon) \subset U$ (because $D_n(z)$ has diameter $\sqrt{d}2^{-n}$). Since for every n there exists some dyadic cube of scale n that contains x (because $\mathbb{R}^d = \bigcup_{z\in\mathbb{Z}^d} D_n(z)$ for every fixed n), we get that for some large enough n there exists $z \in \mathbb{Z}^d$ with $(z,n) \in \Gamma$ and $x \in D_n(z)$. Thus, there must be $(z,n) \in \Lambda$ such that $x \in D_n(z)$ (by just taking the cube of minimal scale containing x). Since this holds for all $x \in U$ we get that

$$U \subset \bigcup_{(z,n)\in\Lambda} \overline{D_n(z)}.$$

This is of course a countable union. Also, since Λ is the indices of maximal dyadic cubes, any two cannot intersect except for the boundary; that is, they are almost disjoint.

□

This construction has a remarkable consequence.

► **Exercise 4.1.** Show that if $U = \bigcup_n B_n$ where $(B_n)_n$ are almost disjoint boxes then $m^*(U) = \sum_n |B_n| = J_*(U)$.

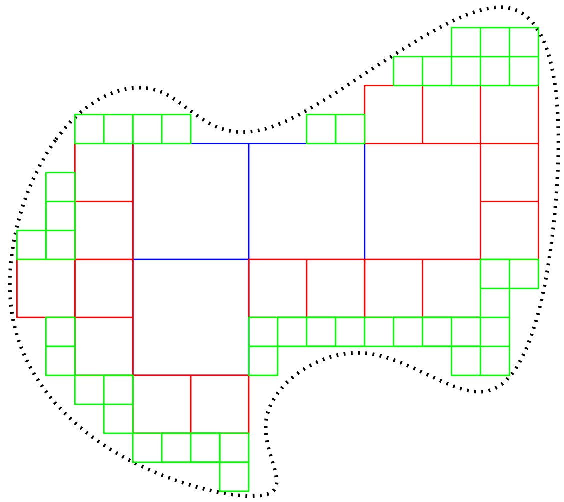

FIGURE 2. Part of the dyadic decomposition of a set. If the set is open, one can continue to capture all points in the set by taking smaller and smaller squares.

♣ **Solution to ex:4.1.** :(
$m^*(U) \leq \sum_n |B_n|$ so it suffices to show that $\sum_n |B_n| \leq J_*(U)$. Note that $\sum_{n=1}^m |B_n| \leq J_*(U)$ for all m because $B_1 \cup \cdots \cup B_m \subset U$. Taking $m \to \infty$ we have $\sum_n |B_n| \leq J_*(U)$.
:) ✓

► **Exercise 4.2.** Show that if $(B_n)_n$ are almost disjoint and $(B'_n)_n$ are almost disjoint, and if $\bigcup_n B_n = \bigcup_n B'_n$ then

$$\sum_n |B_n| = \sum_n |B'_n|.$$

• **Proposition 4.4** (Outer regularity, Lebesgue measure)**.** *For any set A we have*

$$m^*(A) = \inf_{A \subset U \text{ open}} m^*(U).$$

Proof. One direction $m^*(A) \leq m^*(U)$ for any open $U \supset A$, is just monotonicity.

For the other direction, if $m^*(A) = \infty$ there is nothing to prove. Assume $m^*(A) < \infty$. Fix $\varepsilon > 0$. Let $(B_n)_n$ be boxes such that $A \subset \bigcup_n B_n$ and $\sum_n |B_n| \leq m^*(A) + \varepsilon$. For every n let B'_n be an *open* box such that $B_n \subset B'_n$ and $|B'_n| \leq |B_n| + \varepsilon 2^{-n}$. Thus, the set $U = \bigcup_n B'_n$ is an open set containing A and

$$m^*(U) \leq \sum_n |B'_n| \leq m^*(A) + 2\varepsilon.$$

Taking infimum over the left hand side and $\varepsilon \to 0$ completes the proof. □

► **Exercise 4.3.** Show that it is false that

$$m^*(A) = \sup_{A \supset U \text{ open}} m^*(U).$$

• **Proposition 4.5** (Lebesgue measurable sets)**.** *Examples of Lebesgue measurable sets:*

- *If U is open it is Lebesgue measurable.*
- *If $m^*(A) = 0$ then A is Lebesgue measurable.*
- *$\emptyset$ is Lebesgue measurable.*
- *If $(A_n)_n$ is a sequence of Lebesgue measurable sets, then $\bigcup_n A_n$ is Lebesgue measurable.*
- *Any closed set F is Lebesgue measurable.*
- *If A is Lebesgue measurable then so is $A^c = \mathbb{R}^d \setminus A$.*
- *If $(A_n)_n$ is a sequence of Lebesgue measurable sets, then $\bigcap_n A_n$ is Lebesgue measurable.*

Proof. For open sets this is obvious by definition.

If $m^*(A) = 0$ then by outer regularity for any $\varepsilon > 0$ there exists an open set $U \supset A$ such that $m^*(U \setminus A) \leq m^*(U) < m^*(A) + \varepsilon = \varepsilon$. So A is Lebesgue measurable.

$\emptyset$ has $m^*(\emptyset) \leq J^*(\emptyset) = 0$.

Countable unions. Now, if $(A_n)_n$ are all Lebesgue measurable, then: Fix $\varepsilon > 0$. For every n there exists an open set $U_n \supset A_n$ such that $m^*(U_n \setminus A_n) < \varepsilon 2^{-n}$. Thus, $U := \bigcup_n U_n$ is an open set containing $\bigcup_n A_n$ such that by subadditivity,

$$m^*(U \setminus \bigcup_n A_n) \leq \sum_n \mu^*(U_n \setminus A_n) \leq \varepsilon.$$

Since this holds for all $\varepsilon > 0$ we get that $\bigcup_n A_n$ is Lebesgue measurable.

Closed sets. Let F be a closed set. Note that $F = \bigcup_n (F \cap B[0,n])$ where $B[0,n]$ is the closed ball of radius n. So it suffices to prove that F is Lebesgue measurable for closed and bounded F (because then $F \cap B[0,n]$ is Lebesgue measurable for all n). Heine-Borel guaranties then that F is compact.

Fix $\varepsilon > 0$. Let U be an open set such that $F \subset U$ and $m^*(U) \leq m^*(F) + \varepsilon$. The set $U \setminus F$ is open, so we can write $U \setminus F = \bigcup_n B_n$ where $(B_n)_n$ are almost disjoint and closed boxes. For any $m > 0$, the set $C_m := B_1 \cup \cdots \cup B_m$ is closed and disjoint from F. Also, F is compact, so we have additivity:

$$m^*(F) + m^*(C_m) = m^*(F \cup C_m) \leq m^*(F \cup (U \setminus F)) = m^*(U) \leq m^*(F) + \varepsilon.$$

Since F is bounded, we have that $m^*(F) < \infty$, so we conclude that for all $m > 0$,

$$\sum_{n=1}^{m} |B_n| = m^*(B_1 \cup \cdots \cup B_m) \leq \varepsilon,$$

where the equality is because $(B_n)_n$ are almost disjoint. Thus, taking $m \to \infty$,

$$m^*(U \setminus F) = \sum_n |B_n| \leq \varepsilon.$$

This holds for all $\varepsilon > 0$, so F is Lebesgue measurable.

Complements. Now, if A is Lebesgue measurable: For every n let $U_n \supset A$ be an open set such that $m^*(U_n \setminus A) < 2^{-n}$. Let $F_n = U_n^c$ and let $F = \bigcup_n F_n$. Since F_n are closed they are Lebesgue measurable, and thus F is Lebesgue measurable as a countable union. Note that $m^*(A^c \setminus F_n) = m^*(U_n \setminus A) < 2^{-n}$ for all n. Since $A^c \setminus F \subset A^c \setminus F_n$,

$$m^*(A^c \setminus F) \leq \inf_n m^*(A^c \setminus F_n) = 0.$$

Thus, $A^c = F \cup (A^c \setminus F)$ which is the union of two Lebesgue measurable sets, and thus Lebesgue measurable itself.

Countable intersections. If $(A_n)_n$ are all Lebesgue measurable, then so are $(A_n^c)_n$ and thus also

$$\bigcap_n A_n = \left(\bigcup_n A_n^c\right)^c.$$

□

► **Exercise 4.4.** Show that the following are equivalent.

(1) A is Lebesgue measurable.

(2) For every $\varepsilon > 0$ there exists an open set $U \supset A$ such that $m^*(U \setminus A) < \varepsilon$.

(3) For every $\varepsilon > 0$ there exists an open set U such that $m^*(U \triangle A) < \varepsilon$.

(4) For every $\varepsilon > 0$ there exists a closed set F such that $m^*(A \triangle F) < \varepsilon$.

(5) For every $\varepsilon > 0$ there exists a closed set $F \subset A$ such that $m^*(A \setminus F) < \varepsilon$.

(6) For every $\varepsilon > 0$ there exists a Lebesgue measurable set C such that $m^*(A \triangle C) < \varepsilon$.

♣ **Solution to ex:4.4.** :(

(1) $\iff$ (2) is the definition.

(2) $\Rightarrow$ (3) follows by taking the same open set $U \supset A$.

(3) $\Rightarrow$ (2): Fix $\varepsilon > 0$ and let U be such that $m^*(U \triangle A) < \varepsilon$. Let $V \supset U \triangle A$ be an open set such that $m^*(V) < 2\varepsilon$, by outer regularity. Note that $A \subset U \cup (A \setminus U) \subset U \cup V$, which is an open set, and

$$m^*((U \cup V) \setminus A) \leq m^*(U \setminus A) + m^*(V \setminus A) \leq m^*(U \triangle A) + m^*(V) < 3\varepsilon.$$

So (1) $\iff$ (2) $\iff$ (3).

(1) $\Rightarrow$ (4): A is Lebesgue measurable, so also A^c is. Fix $\varepsilon > 0$ and let U be an open set such that $m^*(U \triangle A^c) < \varepsilon$. Let $F = U^c$. So $m^*(F \triangle A) = m^*(U \triangle A^c) < \varepsilon$, and F is closed.

(4) $\Rightarrow$ (5): Fix $\varepsilon > 0$ and let F be a closed set such that $m^*(A \triangle F) < \varepsilon$. By outer regularity let $U \supset A \triangle F$ be an open set such that $m^*(U) < 2\varepsilon$. Then $F \cap U^c$ is a closed

set and $F \cap U^c \subset F \cap A \subset A$, and

$$m^*(A \setminus F \cap U^c) \leq m^*(A \setminus F) + m^*(A \setminus U^c) \leq m^*(A \triangle F) + m^*(U) < 3\varepsilon.$$

(5) $\Rightarrow$ (6): Fix $\varepsilon > 0$ and let $F \subset A$ be a closed set such that $m^*(A \setminus F) < \varepsilon$. The $C = F$ is Lebesgue measurable, and $m^*(A \triangle C) = m^*(A \setminus F) < \varepsilon$.

(6) $\Rightarrow$ (3): Fix $\varepsilon > 0$ and let C be Lebesgue measurable such that $m^*(A \triangle C) < \varepsilon$. Let $U \supset C$ be an open set such that $m^*(U \triangle C) = m^*(U \setminus C) < \varepsilon$. Note that $A \triangle U \subset A \triangle C \bigcup C \triangle U$, so

$$m^*(A \triangle U) \leq m^*(A \triangle C) + m^*(C \triangle U) < 2\varepsilon.$$

:) ✓

► **Exercise 4.5.** Show that if A is Jordan measurable then it is also Lebesgue measurable.

• **Definition 4.6.** A family of subsets $\mathcal{F}$ is a **σ-algebra** if

- $\emptyset \in \mathcal{F}$;
- $\mathcal{F}$ is closed under complements, *i.e.* $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$;
- $\mathcal{F}$ is closed under countable unions, *i.e.* $(A_n)_n \subset \mathcal{F}$ implies $\bigcup_n A_n \in \mathcal{F}$.

► **Exercise 4.6.** Show that if $\mathcal{F}$ is a σ-algebra it is closed under countable intersections, set difference and symmetric difference.

► **Exercise 4.7.** Show that the family of Lebesgue measurable sets form a σ-algebra.

• **Proposition 4.7** (Measure axioms)**.** *Let $\mathcal{L}$ be the family of all Lebesgue measurable sets. Recall that for $A \in \mathcal{L}$ we define $m(A) = m^*(A)$. We the have that $m : \mathcal{F} \to [0, \infty]$ with the following properties:*

- $m(\emptyset) = 0$*;*
- *(Countable additivity) If $(A_n)_n \subset \mathcal{L}$ are pairwise disjoint Lebesgue measurable sets then*

$$m(\biguplus_n A_n) = \sum_n m(A_n).$$

Proof. We only need to prove the assertion regarding additivity.

Case I. $(A_n)_n$ are all compact. Since m^* is additive on disjoint compact sets, for any N,

$$\sum_{n=1}^N m(A_n) = m(\biguplus_{n=1}^N A_n) \leq m(\biguplus_n A_n) \leq \sum_n m(A_n),$$

where the last two inequalities are monotonicity and countable subadditivity. Taking $N \to \infty$ we get the compact case.

Case II. $(A_n)_n$ are all bounded. Fix $\varepsilon > 0$. For every n let F_n be a closed set with $F_n \subset A_n$ and $m^*(A_n \setminus F_n) < \varepsilon 2^{-n}$. Since $F_n \subset A_n$ is bounded, F_n is compact. By subadditivity, $m(A_n) \leq m(F_n) + \varepsilon 2^{-n}$, so

$$\sum_n m(A_n) \leq \sum_n m(F_n) + \varepsilon = m(\biguplus_n F_n) + \varepsilon \leq m(\biguplus_n A_n) + \varepsilon.$$

Taking $\varepsilon \to 0$ completes the bounded case.

Case III. Now for the general case. Let $C_k = B(0, k) \setminus B(0, k-1)$ for all n, and set $D_{n,k} = A_n \cap C_k$. So $(D_{n,k})_{n,k}$ is a sequence of pairwise disjoint bounded Lebesgue measurable sets. Thus, for every n,

$$m(A_n) = m(\biguplus_k D_{n,k}) = \sum_k m(D_{n,k}),$$

and

$$m(\biguplus_n A_n) = m(\biguplus_{n,k} D_{n,k}) = \sum_{n,k} m(D_{n,k}) = \sum_n m(A_n).$$

□

▶ **Exercise 4.8.** Show that if A is Lebesgue measurable then

$$m(A) = \sup_{A \supset K \text{ compact}} m(K).$$

♣ **Solution to ex:4.8.** :(

Monotonicity implies that it suffices to prove that $m(A) \leq \sup_{A \supset K \text{ compact}} m(K)$.

First suppose that A is bounded, so $m(A) < \infty$.

Fix $\varepsilon > 0$. Since A is Lebesgue measurable there exists a closed set $K \subset A$ such that $\mu^*(A \setminus K) < \varepsilon$. Since $K \subset A$ and A is bounded, we have that K is bounded as well, so K is compact because it is bounded and closed. So K is a compact set such that $K \subset A$ and $\mu(K) = \mu(A) - \mu(A \setminus K) \geq \mu(A) - \varepsilon$. This holds for all $\varepsilon > 0$ establishing the claim in the case that A is bounded.

For the case where A is unbounded, consider $A_n = A \cap (B(0,n) \setminus B(0,n-1))$ for all n. These are all bounded, so for every $\varepsilon > 0$ and every n there exists a compact set $K_n \subset A_n$ such that $m(K_n) \geq m(A_n) - \varepsilon 2^{-n}$. Also, $(A_n)_n$ are disjoint so $m(A) = \sum_n m(A_n)$. Thus, if $K'_n = \biguplus_{j=1}^n K_j$, then K'_n is compact, $K'_n \subset A$ and

$$m(K'_n) = \sum_{j=1}^n m(K_j) \geq \sum_{j=1}^n m(A_j) - \varepsilon \to m(A) - \varepsilon.$$

This implies that there exists n large enough so that $m(K'_n) \geq m(A) - 2\varepsilon$.

Conclusion, for any $\varepsilon > 0$ there exists a compact $K \subset A$ such that $m(K) \geq m(A) - \varepsilon$. This proves the unbounded case. :) ✓

Finally, let us end with the following criterion for Lebesgue measurability.

• **Proposition 4.8** (Charathéodory criterion)**.** *The following are equivalent.*

- *A is Lebesgue measurable.*
- *For every box B we have $|B| = m^*(B \cap A) + m^*(B \setminus A)$.*
- *For every elementary set E we have $m(E) = m^*(E \cap A) + m^*(E \setminus A)$.*

Proof. If A is Lebesgue measurable and E is elementary, then $E = (E\cap A)\uplus(E\setminus A)$ which are both Lebesgue measurable sets, so by additivity, $m(E) = m^*(E \cap A) + m^*(E \setminus A)$.

Suppose the second bullet holds. Let E be any elementary set, and write $E = \biguplus_{j=1}^{n} B_j$ for disjoint boxes $(B_j)_{j=1}^{n}$. Then by subadditivity,

$$m^*(E \cap A) + m^*(E \setminus A) \leq \sum_{j=1}^{n} m^*(B_j \cap A) + m^*(B_j \setminus A) = \sum_{j=1}^{n} |B_j| = m(E).$$

Since $m(E) \leq m^*(E \cap A) + m^*(E \setminus A)$ for any set E just by subadditivity, we conclude that equality holds for all elementary sets, proving the third bullet.

Now assume the third bullet. We want to show that A is Lebesgue measurable.

Assume first that $m^*(A) < \infty$. Fix $\varepsilon > 0$. Let $(B_n)_n$ be disjoint boxes such that $\sum_n |B_n| \leq m^*(A) + \varepsilon$ and $A \subset \bigcup_n B_n$ (this exists by the definition of m^*). For every n let $B_n' \supset B_n$ be an open box such that $|B_n'| \leq |B_n| + \varepsilon 2^{-n}$. Note that $U = \bigcup_n B_n'$ is an open set and $U \supset A$. For every n we have that

$$m^*(B_n' \cap A) + m^*(B_n' \setminus A) = |B_n'| \leq |B_n| + \varepsilon 2^{-n}.$$

By subadditivity,

$$m^*(A) + m^*(U \setminus A) \leq \sum_n m^*(B_n' \cap A) + m^*(B_n' \setminus A) \leq \sum_n |B_n| + \varepsilon \leq m^*(A) + 2\varepsilon.$$

Thus, $m^*(U \setminus A) \leq 2\varepsilon$. Since this holds for all $\varepsilon > 0$ this completes the proof for A with $m^*(A) < \infty$.

Now assume that $m^*(A) = \infty$. Let E be any elementary set, and let B be a box. Since $E \cap B$ is elementary, and since $E \setminus (A \cap B) \subseteq (E \setminus B) \cup ((E \cap B) \setminus A)$, we have that

$$\begin{aligned} m^*(E \cap A \cap B) + m^*(E \setminus (A \cap B)) &\leq m^*(E \cap B \cap A) + m^*((E \cap B) \setminus A) + m(E \setminus B) \\ &= m(E \cap B) + m(E \setminus B) = m(E). \end{aligned}$$

Thus, for any elementary E we have the Carathéodory criterion for $A \cap B$,

$$m(E) = m^*(E \cap (A \cap B)) + m^*(E \setminus (A \cap B)).$$

Since B is bounded, we have $m^*(A\cap B) < \infty$ and by the previous part $A\cap B$ is Lebesgue measurable. Since $A = \bigcup_n (A \cap B_n)$ where B_n are boxes of side length n around 0, we have that A is Lebesgue measurable as a countable union of Lebesgue measurable sets.

□

▶ **Exercise 4.9.** For a bounded set A define the **Lebesgue inner measure** by

$$m_*(A) = m(E) - m^*(E \setminus A),$$

for a Lebesgue mesurable set E such that $A \subset E$ and $m(E) < \infty$.

Show that this definition does not depend on the choice of the set E.

Show that $m_*(A) \leq m^*(A)$ and equality holds if and only if A is Lebesgue measurable.

♣ **Solution to ex:4.9.** :(

Let E, F be Lebesgue measurable sets such that $A \subset F \subset E$ and $m(E) < \infty$.

Fix $\varepsilon > 0$. Let $(B_n)_n$ be a sequence of boxes such that $E \setminus A \subset \bigcup_n A_n$ and $\sum_n |B_n| \leq m^*(E \setminus A) + \varepsilon$. Note that

$$F \setminus A \subset (E \setminus A) \setminus (E \setminus F) \subset \bigcup_n (B_n \setminus (E \setminus F)) \qquad \text{and} \qquad E \setminus F \subset \bigcup_n (B_n \cap (E \setminus F)).$$

So,

$$\begin{aligned} m(E \setminus F) + m^*(F \setminus A) &\leq \sum_n m^*(B_n \cap (E \setminus F)^c) + m^*(B_n \cap (E \setminus F)) \\ &= \sum_n |B_n| \leq m^*(E \setminus A) + \varepsilon. \end{aligned}$$

Taking $\varepsilon \to 0$ and rearranging we have that

$$m(E) - m^*(E \setminus A) \leq m(F) - m^*(F \setminus A).$$

Since $E \setminus A \subset (E \setminus F) \cup (F \setminus A)$ we have

$$m(E) - m^*(E \setminus A) \geq m(E) - m(E) + m(F) - m^*(F \setminus A).$$

So we have show that

$$m(E) - m^*(E \setminus A) = m(F) - m(F \setminus A)$$

whenever $A \subset F \subset E$.

For general Lebesgue measurable sets E, F such that $A \subset F$ and $A \subset E$ and $m(E) < \infty, m(F) < \infty$, we have that $A \subset E \cap F$, so

$$m(E) - m^*(E \setminus A) = m(E \cap F) - m^*((E \cap F) \setminus A) = m(F) - m^*(F \setminus A).$$

This shows that m_* is well defined.

Now, if A is Lebesgue measurable then for any Lebesgue measurable set $E \supset A$ with $m(E) < \infty$ we have that $m(E \setminus A) = m(E) - m(A)$. So $m_*(A) = m(E) - m(E \setminus A) = m(A) = m^*(A)$.

On the other hand, assume that $m^*(A) = m_*(A)$. Fix $\varepsilon > 0$ and let $U \supset A$ be an open set such that $A \subset U$ and $m(U) \leq m^*(A) + \varepsilon$. Note that

$$m(U) - \varepsilon \leq m^*(A) = m_*(A) = m(U) - m^*(U \setminus A),$$

so $m^*(U \setminus A) \leq \varepsilon$. :) ✓

► **Exercise 4.10.** Reminder: A G_δ set is a set $A = \bigcap_n U_n$ where $(U_n)_n$ are all open (*i.e.* a countable intersection of open sets). A F_σ set is a set $A = \bigcup_n F_n$ where $(F_n)_n$ are all closed (*i.e.* a countable union of closed sets).

A **null** set is a set with Lebesgue (outer) measure 0.

Show that the following are equivalent.

- A is Lebesgue measurable.
- $A = G \setminus N$ is where G is G_δ and N is null.
- $A = F \cup N$ where F is F_σ and N is null.

► **Exercise 4.11.** Show that if A is Lebesgue measurable then $A+x$ is also Lebesgue measurable and $m(A+x) = m(A)$.

▶ **Exercise 4.12.** Let $\mathcal{L}$ be the family of Lebesgue measurable sets. Suppose that $m' : \mathcal{L} \to [0,\infty]$ admits the following properties:

- $m'(\emptyset) = 0$;
- If $(A_n)_n \subset \mathcal{L}$ are pairwise disjoint Lebesgue measurable sets then $m'(\biguplus_n A_n) = \sum_n m'(A_n)$;
- $m'(A + x) = m'(A)$ for all $A \in \mathcal{L}, x \in \mathbb{R}^d$;
- $m'([0,1]^d) = 1$.

Show that m' is Lebesgue measure.

♣ **Solution to ex:4.12.** :(

We already know that $m'(E) = m(E)$ for any Jordan measurable set E, and specifically for boxes.

It is simple to prove that m' is sub-additive and monotone.

Step I. We show that for $A \in \mathcal{L}$, if $m(A) = 0$ then $m'(A) = 0$: Let A be a Lebesgue measurable set of 0 measure, $m(A) = 0$. For any $\varepsilon > 0$, let $(B_n)_n$ be a sequence of boxes such that $A \subset \bigcup_n B_n$ and such that $\sum_n |B_n| \leq \varepsilon$. Then, $m'(A) \leq \sum_n m'(B_n) = \sum_n |B_n| \leq \varepsilon$, and taking $\varepsilon \to 0$, we have that $m'(A) = 0$.

Step II. We show that if U is open of finite Lebesgue measure, then $m'(U) = m(U)$: Let U be an open set such that $m(U) < \infty$. Write $U = \bigcup_n B_n$ where $(B_n)_n$ are almost disjoint closed boxes. Let $B'_n = (B_n)^\circ$, so $(B'_n)_n$ are pairwise disjoint. Let $F = \bigcup_n B_n \setminus \biguplus_n B'_n$. Since

$$m(F) = m(U) - m(\biguplus_n B'_n) = \sum_n |B_n| - \sum_n |B'_n| = 0,$$

we have that $m'(F) = 0$. Note that $U = F \uplus \biguplus_n B'_n$, so

$$m'(U) = m'(F) + \sum_n m'(B'_n) = \sum_n |B'_n| = m(U).$$

Step III. We show that for any $A \in \mathcal{L}$, the inequality $m'(A) \le m(A)$ holds: Let A be a Lebesgue measurable set with $m(A) < \infty$. Fix $\varepsilon > 0$ and let U be an open set such that $A \subset U$ and $m(U) - m(A) = m(U \setminus A) < \varepsilon$. Then, $m'(A) \le m'(U) \le m(A) + \varepsilon$. Taking $\varepsilon \to 0$ we have that for any $A \in \mathcal{L}$, the inequality $m'(A) \le m(A)$ holds. (The case where $m(A) = \infty$ is immediate.)

Step IV. We show that for any bounded set $A \in \mathcal{L}$ we have $m'(A) = m(A)$: Let $A \in \mathcal{L}$ be a bounded set. Let B be a box bounding $A \subset B$. We have $m'(B \setminus A) = m'(B) - m'(A)$ by additivity of m'. Also,

$$m'(A) \le m(A) = m(B) - m(B \setminus A) \le m'(B) - m'(B \setminus A) = m'(A).$$

Step V. We show that for any $A \in \mathcal{L}$ we have $m'(A) = m(A)$: Let $A \in \mathcal{L}$. Write $A = \biguplus A_n$ where $A_n = B_n \setminus B_{n-1}$, and $B_n = [-n, n]^d$. Then by additivity of both m, m' and since A_n are all bounded,

$$m'(A) = \sum_n m'(A_n) = \sum_n m(A_n) = m(A).$$

:) ✓

Number of exercises in lecture: 12

Total number of exercises until here: 38

Measure Theory

Ariel Yadin

Lecture 5: Abstract measures

We now review the construction of Lebesgue measure, isolating the main abstract properties, in order to generalize it to other settings.

5.1. σ-algebras

We saw in the discussion of Lebesgue measure that the family of Lebesgue measurable sets form a special structure called a σ-algebra.

• **Definition 5.1** (σ-algebra)**.** Let X be any set. We denote by $2^X = \mathcal{P}(X) = \{A : A \subset X\}$ the set of all subsets of X.

A family $\mathcal{F} \subset 2^X$ is called a **σ-algebra** (on X) if:

- $\emptyset \in \mathcal{F}$;
- $\mathcal{F}$ is closed under complements, *i.e.* $A \in \mathcal{F}$ implies $X \setminus A \in \mathcal{F}$;
- $\mathcal{F}$ is closed under countable unions, *i.e.* if $(A_n)_n$ is a sequence in $\mathcal{F}$ then $\bigcup_n A_n \in \mathcal{F}$.

► **Exercise 5.1.** Show that if $\mathcal{F}$ is a σ-algebra on X then:

- $\mathcal{F}$ is closed under countable intersections, *i.e.* if $(A_n)_n$ is a sequence in $\mathcal{F}$ then $\bigcap_n A_n \in \mathcal{F}$.
- $X \in \mathcal{F}$.
- $\mathcal{F}$ is closed under finite unions and finite intersections.
- $\mathcal{F}$ is closed under set differences.
- $\mathcal{F}$ is closed under symmetric differences.

► **Exercise 5.2.** Suppose $\mathcal{F} \subset 2^X$ is a family of subsets satisfying the following:

- $\emptyset \in \mathcal{F}$;
- $\mathcal{F}$ is closed under complements;
- $\mathcal{F}$ is closed under countable intersections.

Show that $\mathcal{F}$ is a σ-algebra.

► **Exercise 5.3.** Show that if $(\mathcal{F}_\alpha)_{\alpha \in I}$ is a collection of σ-algebras on X, then $\bigcap_\alpha F_\alpha$ is also a σ-algebra on X.

• **Proposition 5.2** (σ-algebra generated by subsets)**.** *Let $\mathcal{K}$ be a collection of subsets of X.*

There exists a σ-algebra, denoted $\sigma(\mathcal{K})$ such that $\mathcal{K} \subset \sigma(\mathcal{K})$ and for every other σ-algebra $\mathcal{F}$ such that $\mathcal{K} \subset \mathcal{F}$ we have that $\sigma(\mathcal{K}) \subset \mathcal{F}$.

That is, $\sigma(\mathcal{K})$ is the smallest σ-algebra containing $\mathcal{K}$.

We call $\sigma(\mathcal{K})$ the σ-algebra **generated by** *$\mathcal{K}$.*

Proof. Define

$$\sigma(\mathcal{K}) := \bigcap \{\mathcal{F} \ : \ \mathcal{F} \text{ is a } \sigma\text{-algebra on } X \ , \ \mathcal{K} \subset \mathcal{F}\} \, .$$

This is a σ-algebra with the required properties. □

► **Exercise 5.4.** Show that if $\mathcal{K} \subset \mathcal{L}$ then $\sigma(\mathcal{K}) \subset \sigma(\mathcal{L})$. Also, if $\mathcal{K} \subset \mathcal{F}$ and $\mathcal{F}$ is a σ-algebra, then $\sigma(\mathcal{K}) \subset \mathcal{F}$.

• **Definition 5.3** (Borel σ-algebra)**.** Given a topological space X, the **Borel σ-algebra** is the σ-algebra generated by the open sets. It is denoted $\mathcal{B}(X)$.

Specifically in the case $X = \mathbb{R}^d$ we have that

$$\mathcal{B} = \mathcal{B}_d = \mathcal{B}(\mathbb{R}^d) = \sigma(U \ : \ U \text{ is an open set }).$$

✓ A Borel-measurable set, *i.e.* a set in $\mathcal{B}(X)$, is called a **Borel** set.

▶ **Exercise 5.5.** Prove that

$$\mathcal{B} = \mathcal{B}_d = \mathcal{B}(\mathbb{R}^d) = \sigma(B \ : \ B \text{ is an open box }) = \sigma(B' \ : \ B' \text{ is a closed box }).$$

♣ **Solution to ex:5.5.** :(

If B is an open box then $B \in \mathcal{B}$ so $\sigma(B \ : \ B \text{ is an opend box }) \subset \mathcal{B}$.

If U is an open set then it is a countable union of closed boxes, so $U \in \sigma(B \ : \ B \text{ is a closed box })$, which shows that $\mathcal{B} \subset \sigma(B' \ : \ B' \text{ is a closed box })$.

If $B' = [a_1, b_1] \times \cdots \times [a_d, b_d]$ is a closed box, then for

$$B_n = (a_1 - \tfrac{1}{n}, b_1 + \tfrac{1}{n}) \times \cdots \times (a_d - \tfrac{1}{n}, b_d + \tfrac{1}{n}),$$

we have that $B' = \bigcap_n B_n$ and B_n are all open boxes. Thus, $B' \in \sigma(B \ : \ B \text{ is an open box })$, which implies that $\sigma(B' \ : \ B' \text{ is a closed box }) \subset \sigma(B \ : \ B \text{ is an open box })$. :) ✓

▶ **Exercise 5.6.** Show that

$$\begin{aligned}\mathcal{B}(\mathbb{R}) &= \sigma((a,b)\ :\ -\infty < a < b < \infty) = \sigma([a,b]\ :\ -\infty < a < b < \infty)\\ &= \sigma((a,b]\ :\ -\infty < a < b < \infty) = \sigma([a,b)\ :\ -\infty < a < b < \infty)\\ &= \sigma((a,\infty)\ :\ -\infty < a < \infty) = \sigma([a,\infty)\ :\ -\infty < a < \infty)\\ &= \sigma((-\infty,a)\ :\ -\infty < a < \infty) = \sigma((-\infty,a]\ :\ -\infty < a < \infty).\end{aligned}$$

5.2. Measures

• **Definition 5.4.** A pair $(X,\mathcal{F})$ where $\mathcal{F}$ is a σ-algebra on X is call a **measurable space**. Elements of $\mathcal{F}$ are called **measurable sets**.

Given a measurable space $(X,\mathcal{F})$, a function $\mu : \mathcal{F} \to [0,\infty]$ is called a **measure** (on $(X,\mathcal{F})$) if

- $\mu(\emptyset) = 0$;
- (Additivity) For all sequences $(A_n)_n \subset \mathcal{F}$ of pairwise disjoint sets in $\mathcal{F}$, we have that

$$\mu(\biguplus_n A_n) = \sum_n \mu(A_n).$$

$(X,\mathcal{F},\mu)$ is called a **measure space**.

✓ A measure space $(X,\mathcal{F},\mu)$ is called **finite** if $\mu(X) < \infty$. It is called σ-finite if $X = \bigcup_n A_n$ where $A_n \in \mathcal{F}$ and $\mu(A_n) < \infty$ for all n.

▶ **Exercise 5.7.** Show that any measure is finitely additive; that is, if $(X,\mathcal{F},\mu)$ is a measure space then for any disjoint sets $A,B \in \mathcal{F}$ we have $\mu(A \uplus B) = \mu(A) + \mu(B)$.

Example 5.5. The counting measure: Take $\mathcal{F} = 2^X$ and $\mu(A) = |A|$.

If $f : X \to [0, \infty)$ is a function then

$$\mu(A) := \sup_{(a_n)_n \subset A} \sum_n f(a_n)$$

is a measure on $(X, 2^X)$.

If X is uncountable and

$$\mathcal{F} = \{A \subset X \ : \ A \text{ is countable, or } A^c \text{ is countable } \} ,$$

then $(X, \mathcal{F})$ is a measurable space and

$$\mu(A) = \begin{cases} 1 & A^c \text{ is countable} \\ 0 & A \text{ is countable} \end{cases}$$

is a measure on $(X, \mathcal{F})$. △▽△

► **Exercise 5.8.** Give an example of a set X such that

$$\mu(A) := \begin{cases} \infty & |A| = \infty, \\ 0 & |A| < \infty, \end{cases}$$

is *not* a measure on $(X, 2^X)$.

• **Proposition 5.6** (Basic properties of measures)**.** *Let $(X, \mathcal{F}, \mu)$ be a measure space. Then:*

- *(Monotonicity) For $A \subset B \in \mathcal{F}$ we have $\mu(A) \leq \mu(B)$.*
- *(Subadditivity) If $(A_n)_n \subset \mathcal{F}$ then $\mu(\bigcup_n A_n) \leq \sum_n \mu(A_n)$.*

Proof. Monotonicity follows from $B = A \uplus (B \setminus A)$ and finite additivity, so $\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$.

For subadditivity, let $(A_n)_n \subset \mathcal{F}$ and set

$$B_n = A_n \setminus (\bigcup_{j=1}^{n-1} A_j).$$

So

$$\biguplus_n B_n = \bigcup_n A_n \qquad \text{and} \qquad B_n \subset A_n.$$

Thus,

$$\mu(\bigcup_n A_n) = \mu(\biguplus_n B_n) = \sum_n \mu(B_n) \leq \sum_n \mu(A_n).$$

□

• **Definition 5.7.** For a measure space $(X, \mathcal{F}, \mu)$ a set $N \in \mathcal{F}$ is called a **null** set (or μ-null set) if $\mu(N) = 0$.

A measure space $(X, \mathcal{F}, \mu)$ such that for all null sets $N \in \mathcal{F}$ we have that any $A \subset N$ is also measurable (*i.e.* in $\mathcal{F}$) is called **complete**.

► **Exercise 5.9.** Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $\mathcal{N}$ be the set of all μ-null sets. Define

$$\bar{\mathcal{F}} := \{A \cup F \ : \ A \in \mathcal{F} \ , \ F \subset N \in \mathcal{N}\} .$$

Show that $\bar{\mathcal{F}}$ is a σ-algebra.

Show that if we define $\bar{\mu} : \bar{\mathcal{F}} \to [0, \infty]$ by

$$\bar{\mu}(A \cup F) = \mu(A) \qquad \forall \ \ A \in \mathcal{F}, F \subset N \in \mathcal{N},$$

then $\bar{\mu}$ is a well defined complete measure on $\bar{\mathcal{F}}$; moreover, $\bar{\mu}\big|_{\mathcal{F}} = \mu$ and if ν is a complete measure on $\bar{\mathcal{F}}$ such that $\nu\big|_{\mathcal{F}} = \mu$ then $\nu = \bar{\mu}$.

► **Exercise 5.10.** Show that if $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathcal{F})$ the for any non-negative numbers $a_1, \ldots, a_n$ the function $\mu := \sum_j a_j \mu_j$ is also a measure on $(X, \mathcal{F})$.

5.3. Fatou's Lemma and continuity

• **Proposition 5.8** (Monotone convergence)**.** *Let $(X, \mathcal{F}, \mu)$ be a measure space.*

If $A_1 \subset A_2 \subset \cdots$ is an increasing sequence in $\mathcal{F}$ then

$$\lim_{n\to\infty} \mu(A_n) = \mu(\bigcup_n A_n).$$

If $A_1 \supset A_2 \supset \cdots$ is an decreasing sequence in $\mathcal{F}$ then under the condition that there exists n such that $\mu(A_n) < \infty$ we have that

$$\lim_{n\to\infty} \mu(A_n) = \mu(\bigcap_n A_n).$$

Proof. For the increasing sequence case define $B_n = A_n \backslash A_{n-1}$. Then, $(B_n)_n$ are pairwise disjoint, so

$$\begin{aligned}\lim_{n\to\infty} \mu(A_n) &= \lim_{n\to\infty} \mu(\biguplus_{j=1}^{n} B_j) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(B_j) \\ &= \sum_n \mu(B_n) = \mu(\biguplus_n B_n) = \mu(\bigcup_n A_n).\end{aligned}$$

For the decreasing sequence case, since $\mu(A_k) < \infty$ we define $B_n = A_k \backslash A_n$ (so $B_n = \emptyset$ if $n \leq k$). Note that $\mu(B_n) + \mu(A_n) = \mu(A_k)$ for $n \geq k$ and since $\mu(A_n) \leq \mu(A_k) < \infty$ we have that $\mu(B_n) = \mu(A_k) - \mu(A_n)$. Note that $(B_n)_{n\geq k}$ is an increasing sequence such that $\bigcup_{n\geq k} B_n = A_k \setminus \bigcap_{n\geq k} A_n$. Since $\bigcap_{n\geq k} A_n \subset A_k$ we have

$$\begin{aligned}\mu(A_k) &= \mu(\bigcap_{n\geq k} A_n) + \mu(\bigcup_{n\geq k} B_n) = \mu(\bigcap_n A_n) + \lim_{n\to\infty} \mu(B_n) \\ &= \mu(\bigcap_n A_n) + \lim_{n\to\infty} (\mu(A_k) - \mu(A_n)).\end{aligned}$$

Since $\mu(A_k) < \infty$ we may subtract it from both sides to get the proposition. $\square$

Let $(A_n)_n$ be a sequence of subsets of X. Define

$$\limsup A_n = \bigcap_n \bigcup_{k\geq n} A_k \qquad \text{and} \qquad \liminf A_n = \bigcup_n \bigcap_{k\geq n} A_k.$$

$\limsup A_n$ is the set of all $x \in X$ that appear in infinitely many of the subsets A_n. Similarly, $\liminf A_n$ is the set of all $x \in X$ that appear in all but finitely many of the subsets A_n.

We say that the sequence $(A_n)_n$ converges if $\liminf A_n = \limsup A_n$, and denote this common set by $\lim A_n = \limsup A_n = \liminf A_n$ in this case.

▶ **Exercise 5.11.** Show that $\liminf A_n \subseteq \limsup A_n$.

▶ **Exercise 5.12.** Show that if $(A_n)_n$ is an increasing sequence then $\lim A_n = \bigcup_n A_n$ and if $(A_n)_n$ is a decreasing sequence then $\lim A_n = \bigcap_n A_n$.

• **Lemma 5.9** (Fatou's Lemma)**.** *Let $(X, \mathcal{F}, \mu)$ be a measure space.*

If $(A_n)_n$ is a sequence in $\mathcal{F}$ then

$$\mu(\liminf A_n) \leq \liminf_{n\to\infty} \mu(A_n).$$

If in addition $\mu(\bigcup_n A_n) < \infty$ then

$$\mu(\limsup A_n) \geq \limsup_{n\to\infty} \mu(A_n).$$

Proof. For every n we have that $\bigcap_{k\geq n} A_k \subset A_n$, so

$$\mu(\bigcap_{k\geq n} A_k) \leq \inf_{k\geq n} \mu(A_k).$$

Note that $B_n := \bigcap_{k\geq n} A_k$ for an increasing sequence so

$$\liminf_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \inf_{k\geq n} \mu(A_k) \geq \lim_{n\to\infty} \mu(B_n) = \mu(\bigcup_n B_n) = \mu(\liminf A_n).$$

For $A := \bigcup_n A_n$

$$A \setminus \limsup A_n = A \setminus \bigcap_n \bigcup_{k\geq n} A_k = \bigcup_n \bigcap_{k\geq n} (A \setminus A_k) = \liminf (A \setminus A_n).$$

Thus, if we have $\mu(A) < \infty$ then $\mu(A \setminus \limsup A_n) = \mu(A) - \mu(\limsup A_n)$ and $\mu(A \setminus A_n) = \mu(A) - \mu(A_n)$, so

$$\mu(A)-\mu(\limsup A_n) = \mu(A\setminus\limsup A_n) \leq \liminf_{n\to\infty}(\mu(A)-\mu(A_n)) = \mu(A)-\limsup_{n\to\infty}\mu(A_n),$$

which completes the proof by subtracting $\mu(A)$ from both sides. □

► **Exercise 5.13.** Show that if $\lim A_n$ exists and $\mu(\bigcup_n A_n) < \infty$ then

$$\mu(\lim A_n) = \lim_{n\to\infty}\mu(A_n).$$

Number of exercises in lecture: 13

Total number of exercises until here: 51

Measure Theory

Ariel Yadin

Lecture 6: Outer measures

6.1. Outer measures

• **Definition 6.1.** Let X be any set. An **outer measure** on X is a function $\mu^* : 2^X \to [0,\infty]$ such that

- $\mu^*(\emptyset) = 0$;
- $\mu^*(A) \leq \mu^*(B)$ for all $A \subset B$;
- $\mu^*(\bigcup_n A_n) \leq \sum_n \mu^*(A_n)$.

► **Exercise 6.1.** Show that Lebesgue outer measure m^* is an outer measure (on $\mathbb{R}^d$).

Analogously to the way we defined Lebesgue outer measure, we have:

► **Exercise 6.2.** Let $\mathcal{E} \subset 2^X$ such that $\emptyset \in \mathcal{E}$, and let $\rho : \mathcal{E} \to [0,\infty]$ such that $\rho(\emptyset) = 0$. Define

$$\mu^*(A) := \inf \left\{ \sum_n \rho(E_n) \ : \ A \subset \bigcup_n E_n \ , \ \forall \ n \ , \ E_n \in \mathcal{E} \right\},$$

where $\inf \emptyset = \infty$. Show that μ^* is an outer measure.

♣ **Solution to ex:6.2.** :(

Since $\emptyset \subset \bigcup_n \emptyset$ and $\emptyset \in \mathcal{E}$ we have that $\mu^*(\emptyset) = 0$.

If $A \subset B$ then any sequence $(E_n)_n$ participating in the infimum for $\mu^*(B)$ also participates in the infimum for A. So $\mu^*(A) \leq \mu^*(B)$.

For a sequence $(A_n)_n$: If $\mu^*(A_n) = \infty$ for some n then there is nothing to prove. So assume that $\mu^*(A_n) < \infty$ for all n.

Fix $\varepsilon > 0$ and for every n let $(E_{n,k})_k \subset \mathcal{E}$ be such that $\sum_k \rho(E_{n,k}) \leq \mu^*(A_n) + \varepsilon 2^{-n}$ and $A_n \subset \bigcup_k E_{n,k}$. Then, $\bigcup_n A_n \subset \bigcup_{n,k} E_{n,k}$ and

$$\mu^*(\bigcup_n A_n) \leq \sum_{n,k} \rho(E_{n,k}) \leq \sum_n \mu^*(A_n) + \varepsilon.$$

Taking $\varepsilon \to 0$ completes the proof. :) ✓

Compare this to the case that $\mathcal{E}$ is the collection of boxes in $\mathbb{R}^d$.

6.2. Measurability

Recall the Carathéodory's criterion for Lebesgue measurability: $A \subset \mathbb{R}^d$ is Lebesgue measurable if and only if for every elementary set E we have $m(E) = m^*(E \cap A) + m^*(E \setminus A)$. This motivates the following definition.

• **Definition 6.2** (Measurable sets)**.** Let μ^* be an outer measure on X. A subset $A \subset X$ is called **μ^*-measurable** (or simply measurable) if for every subset $E \subset X$ we have

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

► **Exercise 6.3.** Show that A is μ^*-measurable if and only if for every $E \subset X$ such that $\mu^*(E) < \infty$ we have

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(E).$$

► **Exercise 6.4.** Let m be Lebesgue measure in $\mathbb{R}^d$. We have seen that m^* is an outer measure. Show that A is Lebesgue measurable if and only if it is m^*-measurable.

♣ **Solution to ex:6.4.** :(

If A is m^*-measurable, then we have already seen that A is Lebesgue measurable, since it is enough to require that $m(E) = m^*(E \cap A) + m^*(E \cap A^c)$ for every elementary set E.

Assume that A is Lebesgue measurable. Then $|B| = m^*(B \cap A) + m^*(B \cap A^c)$ for every box B.

Let $E \subset X$ be any subset such that $m^*(E) < \infty$. Fix $\varepsilon > 0$. Let $(B_n)_n$ be pairwise disjoint boxes such that $E \subset \biguplus_n B_n$ and $\sum_n |B_n| \leq m^*(E) + \varepsilon$. $E \cap A \subset \biguplus_n (B_n \cap A)$ and $E \cap A^c \subset \biguplus_n (B_n \cap A^c)$, so

$$m^*(E \cap A) + m^*(E \cap A^c) \leq \sum_n m(B_n \cap A) + m(B_n \cap A^c) = \sum_n |B_n| \leq m^*(E) + \varepsilon.$$

Taking $\varepsilon \to 0$ shows that A is m^*-measurable. :) ✓

••• **Theorem 6.3** (Charathéodory's Theorem)**.** *Let μ^* be an outer measure on X and let $\mathcal{F}$ be the collection of all μ^*-measurable subsets. Then, $\mathcal{F}$ is a σ-algebra, and μ^* restricted to $\mathcal{F}$ is a complete measure on $(X, \mathcal{F})$.*

Proof. First, $\emptyset \in \mathcal{F}$ since $\mu^*(E) = \mu^*(E \cap X) + \mu^*(E \cap \emptyset)$ for all $E \subset X$.

Also, $\mathcal{F}$ is closed under complements, because $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ is a symmetric equation in A, A^c.

Now, if $A, B \in \mathcal{F}$ then $A \cup B = (A \cap B) \uplus (A \cap B^c) \uplus (A^c \cap B)$, so for any E with $\mu^*(E) < \infty$,

$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\
&= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\
&\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c).
\end{aligned}$$

Thus $A \cup B \in \mathcal{F}$ as well.

This shows that $\mathcal{F}$ is closed under *finite* unions.

Note that if $A, B \in \mathcal{F}, A \cap B = \emptyset$ then

$$\mu^*(A \uplus B) = \mu^*((A \uplus B) \cap A) + \mu^*((A \uplus B) \cap A^c) = \mu^*(A) + \mu^*(B).$$

So μ^* is finitely additive on $\mathcal{F}$.

Now, if $(A_n)_n$ are pairwise disjoint sets in $\mathcal{F}$ then for all n,

$$\begin{aligned}\mu^*(E \cap \biguplus_{j=1}^n A_j) &= \mu^*(E \cap \biguplus_{j=1}^n A_j \cap A_n) + \mu^*(E \cap \biguplus_{j=1}^n A_j \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap \biguplus_{j=1}^{n-1} A_j) = \cdots = \sum_{j=1}^n \mu^*(E \cap A_j).\end{aligned}$$

Thus, for any n,

$$\mu^*(E) = \mu^*(E \cap \biguplus_{j=1}^n A_n) + \mu^*(E \setminus \biguplus_{j=1}^n A_n) \geq \sum_{j=1}^n \mu^*(E \cap A_j) + \mu^*(E \setminus \biguplus_n A_n).$$

Taking $n \to \infty$ we have that for $A = \biguplus_n A_n$,

$$\mu^*(E) \geq \sum_n \mu^*(E \cap A_n) + \mu^*(E \cap A^c) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

This implies that $A \in \mathcal{F}$ and that the above inequalities are all equalities, so

$$\mu^*(E \cap A) = \sum_n \mu^*(E \cap A_n).$$

Thus, taking $E = A$ we get that μ^* is countable additive on $\mathcal{F}$ and that $\mathcal{F}$ is closed under countable *disjoint* unions.

So we are only left with showing that $\mathcal{F}$ is closed under countable unions.

Now if $(A_n)_n$ is any sequence in $\mathcal{F}$ (not necessarily disjoint), then set $B_n = A_n \setminus \bigcup_{j=1}^{n-1} A_j$. So $(B_n)_n$ are pairwise disjoint. Because $\mathcal{F}$ is closed under complements and finite unions we have that $B_n \in \mathcal{F}$ for all n. Thus, $\bigcup_n A_n = \biguplus_n B_n \in \mathcal{F}$, which proves that $\mathcal{F}$ is closed under countable unions.

Finally, to show that $(X, \mathcal{F}, \mu^*\big|_{\mathcal{F}})$ is indeed complete: If A admits $\mu^*(A) = 0$ and then by monotonicity, for any E with $\mu^*(E) < \infty$,

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(E \cap A^c) \leq \mu^*(E).$$

So $A \in \mathcal{F}$. $\square$

► **Exercise 6.5.** [Folland p.32 ex.17] Let μ^* be an outer measure on X and let $(A_n)_n$ be a sequence of pairwise disjoint μ^*-measurable subsets. Show that for any $A \subset X$,

$$\mu^*(A \cap (\biguplus_n A_n)) = \sum_n \mu^*(A \cap A_n).$$

6.3. Pre-measures

It is also interesting to what degree this resulting measure is unique.

• **Definition 6.4.** An **algebra** is a collection of subsets $\mathcal{A} \subset 2^X$ that is closed under finite unions and complements.

► **Exercise 6.6.** Let B be a box in $\mathbb{R}^d$. Show that

$$\mathcal{A} = \{ E \subset B \ : \ E \text{ is elementary } \}$$

is an algebra over B.

► **Exercise 6.7.** Show that an algebra $\mathcal{A}$ is closed under finite intersections, set differences, symmetric differences.

Show that $\emptyset \in \mathcal{A}$.

► **Exercise 6.8.** Let $\mathcal{A}$ be an algebra such that for all sequences $(A_n)_n \subset \mathcal{A}$ that are pairwise disjoint we have that $\biguplus_n A_n \in \mathcal{A}$.

Show that $\mathcal{A}$ is a σ-algebra.

• **Definition 6.5** (Pre-measure)**.** Given an algebra $\mathcal{A}$ on X, a **pre-measure** is a function $\mu_0 : \mathcal{A} \to [0, \infty]$ such that $\mu_0(\emptyset) = 0$ and for any sequence $(A_n)_n \subset \mathcal{A}$ that are pairwise disjoint and admit $\biguplus_n A_n \in \mathcal{A}$ we have that $\mu_0(\biguplus_n A_n) = \sum_n \mu_0(A_n)$.

• **Lemma 6.6.** *Let μ_0 be a pre-measure on $\mathcal{A}$ over X. Define*

$$\mu^*(A) := \inf\left\{\sum_n \mu_0(E_n) \ : \ A \subset \bigcup_n E_n \ , \ \forall \ n \ , \ E_n \in \mathcal{A}\right\}.$$

Then μ^* is an outer measure such that $\mu^*\big|_{\mathcal{A}} = \mu_0$ and every set in $\mathcal{A}$ is μ^*-measurable.

Proof. We have already seen that such a definition gives rise to an outer measure.

For any $A \in \mathcal{A}$ let $(E_n)_n$ be a sequence in $\mathcal{A}$ such that $A \subset \bigcup_n E_n$. Let $B_n = A \cap (E_n \setminus \bigcup_{j=1}^{n-1} E_j)$. Then, $B_n \in \mathcal{A}$ for all n and $E_n \setminus B_n \in \mathcal{A}$ for all n. Thus, $\mu_0(E_n) = \mu_0(B_n) + \mu_0(E_n \setminus B_n) \geq \mu_0(B_n)$ for all n, and also because $\biguplus_n B_n = A \in \mathcal{A}$,

$$\mu_0(A) = \sum_n \mu_0(B_n) \leq \sum_n \mu_0(E_n).$$

Taking infimum over all such sequences $(E_n)_n \subset \mathcal{A}$ we obtain that $\mu_0(A) \leq \mu^*(A)$. Since $\mu^*(A) \leq \mu_0(A)$ by definition, we get that $\mu_0(A) = \mu^*(A)$. This completes the first assertion.

For the second assertion, let $A \in \mathcal{A}$ and $E \subset X$ such that $\mu^*(E) < \infty$. Fix $\varepsilon > 0$ and let $(E_n)_n \subset \mathcal{A}$ be such that $E \subset \bigcup_n E_n$ and $\sum_n \mu_0(E_n) \leq \mu^*(E) + \varepsilon$. For all n we have $\mu_0(E_n \cap A) + \mu_0(E_n \cap A^c) = \mu_0(E_n)$. Thus,

$$\mu^*(E) \geq -\varepsilon + \sum_n \mu_0(E_n \cap A) + \sum_n \mu_0(E_n \cap A^c) = -\varepsilon + \sum_n \mu^*(E_n \cap A) + \sum_n \mu^*(E_n \cap A^c)$$
$$\geq -\varepsilon + \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

So A is μ^*-measurable. □

••• **Theorem 6.7** (Charathéodory's Extension Theorem)**.** *Let μ_0 be a pre-measure on an algebra $\mathcal{A}$ over X. Let $\mathcal{F} = \sigma(\mathcal{A})$. Then, there exists a measure μ on $(X, \mathcal{F})$ such that $\mu\big|_{\mathcal{A}} = \mu_0$.*

Also, if ν is a measure on $(X, \mathcal{F})$ such that $\nu\big|_{\mathcal{A}} = \mu_0$ then $\nu(A) \leq \mu(A)$ for all $A \in \mathcal{F}$, and $\nu(A) = \mu(A)$ for all $A \in \mathcal{F}$ with $\mu(A) < \infty$.

Moreover, if $X = \bigcup_n A_n$ for $A_n \in \mathcal{A}, \mu_0(A_n) < \infty$ (i.e. μ_0 is σ-finite), then $\nu = \mu$.

Proof. Lemma 6.6 tells us that μ_0 extends to a measure μ on the μ^*-measurable sets, which form a σ-algebra that contains $\mathcal{A}$, and thus contains $\mathcal{F} = \sigma(\mathcal{A})$. Here

$$\mu^*(A) := \inf \left\{ \sum_n \mu_0(E_n) \ : \ A \subset \bigcup_n E_n \ , \ \forall \ n \ , \ E_n \in \mathcal{A} \right\} .$$

Now, if $A \in \mathcal{F}$, and $(E_n)_n \subset \mathcal{A}$ is such that $A \subset \bigcup_n E_n$, then

$$\nu(A) \leq \sum_n \nu(E_n) = \sum_n \mu_0(E_n).$$

Taking infimum over all such sequences we get that $\nu(A) \leq \mu^*(A) = \mu(A)$ for all $A \in \mathcal{F}$.

If $\mu(A) = \mu^*(A) < \infty$ then for any $\varepsilon > 0$ we may choose $(E_n)_n$ such that $A \subset \bigcup_n E_n$ and $\sum_n \mu_0(E_n) \leq \mu^*(A) + \varepsilon$. Thus, $\mu(\bigcup_n E_n) \leq \mu(A) + \varepsilon$ which implies that $\mu(\bigcup_n E_n \setminus A) \leq \varepsilon$. Hence,

$$\mu(A) \leq \mu(\bigcup_n E_n) = \lim_{n \to \infty} \mu_0(\bigcup_{j=1}^n E_j) = \nu(\bigcup_n E_n)$$
$$\leq \nu(A) + \nu(\bigcup_n E_n \setminus A) \leq \nu(A) + \mu(\bigcup_n E_n \setminus A) \leq \nu(A) + \varepsilon.$$

Thus, taking $\varepsilon \to 0$ implies that $\nu(A) = \mu(A)$.

Finally, if μ_0 is σ-finite, then we may write $X = \biguplus X_n$ for $X_n \in \mathcal{A}$ pairwise disjoint and $\mu_0(X_n) < \infty$. So for any $A \in \mathcal{F}$ we have $A = \biguplus_n (A \cap X_n)$. Since $\mu(A \cap X_n) < \infty$ for all n we get that

$$\mu(A) = \sum_n \mu(A \cap X_n) = \sum_n \nu(A \cap X_n) = \nu(A).$$

□

► **Exercise 6.9.** [Folland, p.32, ex.18] Let $\mathcal{A} \subset 2^X$ be an algebra. Let $\mathcal{A}_\sigma$ be all countable unions of sets in $\mathcal{A}$; let $\mathcal{A}_{\sigma\delta}$ be all countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a pre-measure on $\mathcal{A}$ and let μ^* be the induced outer measure.

(1) Show that for any $E \subset X$ and any $\varepsilon > 0$ there exists $A \in \mathcal{A}_\sigma$ such that $\mu^*(A) \leq \mu^*(E) + \varepsilon$ and $E \subset A$.

(2) Show that if $\mu^*(E) < \infty$, then E is μ^*-measurable if and only if there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$.

(3) Show that if μ_0 is σ-finite then the assumption in (b) above that $\mu^*(E) < \infty$ is superfluous.

♣ **Solution to ex:6.9.** :(

- Fix $\varepsilon > 0$. Let $(A_n)_n$ be a sequence in $\mathcal{A}$ such that $E \subset A := \bigcup_n A_n$ and $\sum_n \mu_0(A_n) \leq \mu^*(E) + \varepsilon$. Then since $\mu^*(A) \leq \sum_n \mu^*(A_n) = \sum_n \mu_0(A_n)$ we are done.
- ($\Leftarrow$) Let $(A_{n,k})_{n,k}$ be sets in $\mathcal{A}$ such that $B = \bigcap_k \bigcup_n A_{n,k} \in \mathcal{A}_{\sigma\delta}$ and such that $E \subset B$ and $\mu^*(B \setminus E) = 0$.

  Let $F \subset X$. Then, since $A_{n,k}$ are all μ^*-measurable, then also B is μ^*-measurable. So,

$$\mu^*(F \cap E) + \mu^*(F \cap E^c) \leq \mu^*(F \cap B) + \mu^*(F \cap E^c \cap B^c) + \mu^*(F \cap E^c \cap B)$$
$$\leq \mu^*(F \cap B) + \mu^*(F \cap B^c) + \mu^*(B \setminus E) = \mu^*(F).$$

  This holds for any F so E is μ^*-measurable. (*) Note: this direction did not require $\mu^*(E) < \infty$.

  ($\Rightarrow$) If E is μ^*-measurable, then for any $n > 0$ there exists $A_n \in \mathcal{A}_\sigma$ such that $E \subset A_n$ and $\mu^*(A_n) \leq \mu^*(E) + \frac{1}{n}$. Thus, for $B = \bigcap_n A_n \in \mathcal{A}_{\sigma\delta}$, since $E \subset B$,

$$\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c) = \mu^*(E) + \mu^*(B \setminus E).$$

  Also, since $B \subset A_n$ for all n,

$$\mu^*(B) \leq \inf_n \mu^*(A_n) \leq \mu^*(E).$$

  Thus, $\mu^*(B \setminus E) \leq 0$.

- The only direction we need to show is ($\Rightarrow$) because the other direction holds without the finite measure assumption.

  If $X = \biguplus_n F_n$ such that $\mu_0(F_n) < \infty$ for all n, then for any $A \subset X$, since F_n are all μ^*-measurable,

  $$\mu^*(A) = \sum_n \mu^*(A \cap F_n).$$

  (This was shown in a previous exercise.)

  For every k we have that $E \cap F_k$ is measurable and has finite outer measure. So let $B_k \subset F_k$ be a $\mathcal{A}_{\sigma\delta}$ set such that $E \cap F_k \subset B_k$ and $\mu^*(B_k \setminus (E \cap F_k)) = 0$. Let $B = \biguplus_k B_k$. We have that $E \subset B$ and $B \cap F_k \cap E^c = B_k \setminus (E \cap F_k)$. So,

  $$\mu^*(B \cap E^c) = \sum_k \mu^*(B \cap F_k \cap E^c) = \sum_k \mu^*(B_k \setminus (E \cap F_k)) = 0.$$

  Thus we are left with showing that $B \in \mathcal{A}_{\sigma\delta}$.

  For every k write

  $$B_k = \bigcap_n \bigcup_j A^k_{n,j},$$

  where $A^k_{n,j} \in \mathcal{A}$ and $A^k_{n,j} \subset F_k$ for all n, j, k. Writing $C_n := \biguplus_k \bigcup_j A^k_{n,j}$ we have that $C_n \in \mathcal{A}_\sigma$. We have that $B_k = \bigcap_n (C_n \cap F_k) = F_k \cap \bigcap_n C_n$, so

  $$B = \biguplus_k B_k = \biguplus_k F_k \cap \bigcap_n C_n = \bigcap_n C_n \in \mathcal{A}_{\sigma\delta}.$$

  :) ✓

► **Exercise 6.10.** [Folland p.32, ex.19] Let μ^* be an outer measure induced from a pre-measure μ_0 on X such that $\mu_0(X) < \infty$. Define $\mu_*(A) = \mu_0(X) - \mu^*(A^c)$. Show that $\mu_*(A) \leq \mu^*(A)$ and that A is measurable if and only if $\mu_*(A) = \mu^*(A)$.

► **Exercise 6.11.** [Folland p.32, ex.23] Let $\mathcal{A}$ be the collection of finite unions of sets of the form $(a, b] \cap \mathbb{Q}$, for all $-\infty \leq a < b \leq \infty$. Show that $\mathcal{A}$ is an algebra over $\mathbb{Q}$.

What is the σ-algebra generated by $\mathcal{A}$?

Define $\mu_0(\emptyset) = 0$ and $\mu_0(A) = \infty$ for $\emptyset \neq A \in \mathcal{A}$. Show that μ_0 is a pre-measure, and that μ_0 can be extended to a measure on $2^{\mathbb{Q}}$ in more than one way.

► **Exercise 6.12.** [Folland p.33] Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let μ^* be the outer measure induced by μ. Let $Y \notin \mathcal{F}$ be such that $\mu^*(Y) = \mu^*(X)$.

Show that if $A, B \in \mathcal{F}$ are such that $A \cap Y = B \cap Y$, then $\mu(A) = \mu(B)$.

Define

$$\mathcal{F}_Y := \{A \cap Y \ : \ A \in \mathcal{F}\} .$$

Show that $\mathcal{F}_Y$ is a σ-algebra on Y.

Define ν on $(Y, \mathcal{F}_Y)$ by $\nu(A \cap Y) = \mu(A)$, which is well defined by the above. Prove that ν is a measure.

Number of exercises in lecture: 12

Total number of exercises until here: 63

Measure Theory

Ariel Yadin

Lecture 7: Lebesgue-Stieltjes Theory

7.1. Lebesgue-Stieltjes measure

Define $\mathcal{A}_1 = \mathcal{A}(\mathbb{R})$ to be the collection of finite disjoint unions of intervals of the form $(a, b], (a, \infty), (-\infty, b], \emptyset$ for $-\infty < a < b < \infty$.

► **Exercise 7.1.** Show that $\mathcal{A}_1$ is an algebra on $\mathbb{R}$ and that $\sigma(\mathcal{A}_1) = \mathcal{B}_1$ (the Borel σ-algebra).

• **Lemma 7.1.** *Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function that is right-continuous (continue á droite). Define $\mu_0(\emptyset) = 0$ and for $-\infty \leq a < b < \infty$, $\mu_0((a, b]) = F(b) - F(a)$. Also, define $\mu((a, \infty)) = F(\infty) - F(a)$. (Here by $F(-\infty)$ and $F(\infty)$ we mean the limits at $-\infty$ and ∞ respectively.) For any $A \in \mathcal{A}_1$ that is a finite disjoint union of such intervals, $A = I_1 \uplus \cdots \uplus I_n$, define $\mu_0(A) = \mu_0(I_1) + \cdots + \mu_0(I_n)$.*

Then, μ_0 is a well defined pre-measure on $\mathcal{A}_1$.

Proof. Let $\mathcal{I}$ denote the collection of all intervals of the form $(a, b], (a, \infty), (-\infty, b], \emptyset$ for $-\infty < a < b < \infty$.

Well defined. To show that μ_0 is well defined: Note that if $I, J \in \mathcal{I}$, then also $I \cap J \in \mathcal{I}$.Now, if $(a, b] = (a_1, b_1] \uplus \cdots \uplus (a_n, b_n]$, then without loss of generality $a = a_1 < b_1 = a_2 < b_2 = \cdots = a_n < b_n = b$. So,

$$F(b) - F(a) = \sum_{j=1}^{n} F(b_j) - F(a_j).$$

Also, a similar identity holds when writing $(-\infty, b]$ or (a, ∞) as finite disjoint unions. Thus, if $A = I_1 \uplus \cdots \uplus I_n = J_1 \uplus \cdots \uplus J_k$ then for all i we have that $I_i = (I_i \cap J_1) \uplus \cdots \uplus$

$(I_i \cap J_k)$, so

$$\mu_0(I_i) = \sum_{j=1}^{k} \mu_0(I_i \cap J_j),$$

and we get that

$$\sum_{i=1}^{n} \mu_0(I_i) = \sum_{i,j} \mu_0(I_i \cap J_j) = \sum_{j=1}^{k} \mu_0(J_j).$$

This shows that μ_0 is well defined.

Finite additivity. This also proves that μ_0 is finitely additive. Indeed, if $A_1, \ldots, A_n$ are pairwise disjoint sets in $\mathcal{A}_1$, then we may write $A_j = \biguplus_{k=1}^{n_j} I_{j,k}$ for $I_{j,k} \in \mathcal{I}$, and then

$$\mu_0(\biguplus_{j=1}^{n} A_j) = \mu_0(\biguplus_{j=1}^{n} \biguplus_{k=1}^{n_j} I_{j,k}) = \sum_{j=1}^{n} \sum_{k=1}^{n_j} \mu_0(I_{j,k}) = \sum_{j=1}^{n} \mu_0(A_j).$$

We will use the finite additivity of μ_0 in what follows.

Pre-measure. To show that it is a pre-measure we need to consider $(A_n)_n \subset \mathcal{A}_1$ that are pairwise disjoint and $\biguplus_n A_n = A \in \mathcal{A}_1$. Since every A_n is a finite disjoint union of intervals, we may assume without loss of generality that A_n is an interval for all n. Since $A = I_1 \uplus \cdots \uplus I_k$ is also a finite disjoint union of intervals, we have that $\biguplus_n (A_n \cap I_j) = I_j$ which is a countable disjoint union of intervals whose union is an interval. So using the additivity of μ_0, we may assume without loss of generality that A is an interval. Also, we may re-order $(A_n)_n$ so that $A_n = (a_n, b_n]$ and $a_n < b_n = a_{n+1}$ for all n. Set $B_n = \biguplus_{j=1}^{n} A_j$.

$A \setminus B_n \in \mathcal{A}_1$, so $\mu_0(A) = \mu_0(B_n) + \mu_0(A \setminus B_n)$. Thus,

$$\mu_0(A) = \mu_0(B_n) + \mu_0(A \setminus B_n) \geq \sum_{j=1}^{n} \mu_0(A_j).$$

Taking $n \to \infty$ we get that $\mu_0(A) \geq \sum_n \mu_0(A_n)$. So we only need to prove the other inequality.

Start with the case $A = (a, b]$ with $-\infty < a < b < \infty$. Fix $\varepsilon > 0$. F was assumed to be right-continuous. So we may choose $\delta > 0$ and $\delta_n > 0$ such that for all n,

$$F(a + \delta) < F(a) + \varepsilon \qquad \text{and} \qquad F(b_n + \delta_n) < F(b_n) + \varepsilon 2^{-n}.$$

We have the open cover $[a + \delta, b] \subset \bigcup_n (a_n, b_n + \delta_n)$, and since $[a + \delta, b]$ is compact, there is a finite sub-cover. That is, there exists $m > 0$ such that

- $[a+\delta, b] \subset \bigcup_{k=1}^m (a_k, b_k+\delta_k)$,
- and for all k, we have $b_k + \delta_k \in (a_{k+1}, b_{k+1}+\delta_{k+1})$.

In this case we get that

$$\begin{aligned}
\mu_0(A) &= F(b) - F(a) \leq F(b) - F(a+\delta) + \varepsilon \leq F(b_m+\delta_m) - F(a_1) + \varepsilon \\
&= F(b_m+\delta_m) - F(a_m) + \sum_{j=1}^{m-1} F(a_{j+1}) - F(a_j) + \varepsilon \\
&\leq F(b_m+\delta_m) - F(a_m) + \sum_{j=1}^{m-1} F(b_j+\delta_j) - F(a_j) + \varepsilon \\
&= \sum_{j=1}^{m-1} F(b_j+\delta_j) - F(b_j) + F(b_j) - F(a_j) + \varepsilon \\
&\leq \sum_{j=1}^{m} \mu_0(A_j) + \sum_{j=1}^{m} \varepsilon 2^{-j} + \varepsilon \\
&\leq \sum_n \mu_0(A_n) + \sum_n \varepsilon 2^{-n} + \varepsilon \leq \sum_n \mu_0(A_n) + 2\varepsilon.
\end{aligned}$$

Taking $\varepsilon \to 0$ completes the case where $A = (a, b]$.

If $A = (-\infty, b]$ then for any large enough $r > 0$ we have that $(-r, b] = \biguplus_n (A_n \cap (-r, b])$. Since $\mu_0(A_n \cap (-r, b]) \leq \mu_0(A_n \cap (-r, b]) + \mu_0(A_n \cap (-r, b]^c) = \mu_0(A_n)$,

$$F(b) - F(-r) = \mu_0((-r, b]) = \sum_n \mu_0(A_n \cap (-r, b]) = \sum_n \mu_0(A_n),$$

and taking $r \to \infty$ we get that

$$\mu_0(A) = F(b) - F(-\infty) \leq \sum_n \mu_0(A_n).$$

If $A = (a, \infty)$, then similarly, for any large enough $r > 0$ we have that $(a, r] = \biguplus_n (A_n \cap (a, r])$, so

$$F(r) - F(a) = \sum_n \mu_0(A_n \cap (a, r]) \leq \sum_n \mu_0(A_n),$$

and taking $r \to \infty$ completes the proof. □

★★★ **THEOREM 7.2** (Lebesgue-Stieltjes measure)**.** *If $F : \mathbb{R} \to \mathbb{R}$ is a right-continuous non-decreasing function, then there exists a unique measure μ_F on $(\mathbb{R}, \mathcal{B})$ such that for* *all $a < b$,*

$$\mu_F((a,b]) = F(b) - F(a).$$

In fact, $\mu_F = \mu_G$ if and only if $F - G$ is constant.

Moreover, one may recover F from μ_F: If μ is a measure on $(\mathbb{R}, \mathcal{B})$ that is finite on all bounded sets in $\mathcal{B}$, if we define

$$F(x) = \begin{cases} \mu((0,x]) & x > 0 \\ 0 & x = 0 \\ -\mu((x,0]) & x < 0 \end{cases}$$

then F is non-decreasing and right-continuous, and $\mu_F = \mu$.

Proof. Lemma 7.1 and Charathódory's Extension Theorem tell us that we may extend the pre-measure $\mu_0((a,b]) = F(b) - F(a)$ uniquely to $(\mathbb{R}, \mathcal{B})$. Also, if $\mu_F = \mu_G$ then $F(b) - F(a) = G(b) - G(a)$ for all $a < b$ which implies that $F = G + F(0) - G(0)$.

If μ is a measure on $(\mathbb{R}, \mathcal{B})$ that is finite on all bounded sets in $\mathcal{B}$, one may check that by the definition of F in the theorem, $\mu((a,b]) = F(b) - F(a)$ for all $a < b$. So F is non-decreasing (since μ is non-negative). F is right continuous from the continuity properties of measures: For any x and $\varepsilon_n \searrow 0$

$$F(x + \varepsilon_n) - F(x) = \mu((x, x + \varepsilon_n]) \to \mu(\bigcap_n (x, x + \varepsilon_n]) = \mu(\emptyset) = 0.$$

□

✓ The measure μ_F is called the **Lebesgue-Stieltjes measure** associated to F.

► **Exercise 7.2.** Show that μ_F satisfies the following properties.

- $\mu_F(\{a\}) = F(a) - F(a-)$.
- $\mu_F((a,b)) = F(b-) - F(a)$.
- $\mu_F([a,b]) = F(b) - F(a-)$.
- $\mu_F([a,b)) = F(b-) - F(a-)$.

▶ **Exercise 7.3.** Show that for $F(x) = x$ the associated measure μ_F is Lebesgue measure on $\mathbb{R}$.

♣ **Solution to ex:7.3.** :(

Note that μ_F is a measure with $\mu((0, 1]) = 1$ and $\mu((a, b] + x) = F(b + x) - F(a + x) = b - a = \mu((a, b])$. So μ is the extension of Jordan measure on elementary sets, and must be Lebesgue measure. :) ✓

▶ **Exercise 7.4.** Let $A \subset \mathbb{R}$ be Lebesgue measurable. Suppose that $m(A) > 0$. Show that for any $0 < \varepsilon < 1$ there exists an *open* interval I such that $m(A \cap I) \geq (1 - \varepsilon)m(I)$.

♣ **Solution to ex:7.4.** :(

First suppose the $0 < m(A) < \infty$. Define

$$\beta = \sup \left\{ \tfrac{m(A \cap I)}{m(I)} \ : \ I \text{ is an open interval} \right\}.$$

Of course $\beta \leq 1$. We need to show that $\beta = 1$.

Fix $\varepsilon > 0$ and let $(I_n)_n$ be *disjoint* intervals (not necessarily open or closed) such that $A \subset \biguplus_n I_n$ and $\sum_n \lambda(I_n) \leq m(A) + \varepsilon$.

Recall that Lebesgue measure of all points is 0. So $m(A \cap I)$ is the same for open, closed, half-open-closed intervals, and similarly for $m(I)$. Thus, $m(A \cap I_n) \leq \beta m(I_n)$ for all n, and we obtain

$$m(A) = \sum_n m(A \cap I_n) \leq \beta \sum_n m(I_n) \leq \beta(m(A) + \varepsilon).$$

Taking $\varepsilon \to 0$ we get that $\beta = 1$.

In the case that $m(A) = \infty$, let N be large enough so that $m(A \cap (-N, N)) > 0$. Since $m(A \cap (-N, N)) \leq 2N < \infty$, by the previous part, for any $0 < \varepsilon < 1$ there exists an open interval I such that

$$m(A \cap I) \geq m(A \cap (-N, N) \cap I) > (1 - \varepsilon)m(I).$$

:) ✓

7.2. Regularity

We have already seen outer and inner regularity for Lebesgue measure. This is a more general fact, for Lebesgue-Stieltjes measures.

• **Proposition 7.3** (Outer and inner regularity). *For any Lebesgue measurable set A we have*

$$\mu_F(A) = \inf_{A \subset U \text{ open}} \mu_F(U) = \sup_{A \supset K \text{ compact}} \mu_F(K).$$

Proof. If $\mu(A) = \infty$ then any $U \supset A$ also admits $\mu(U) = \infty$ so outer regularity is simple in this case. So assume that $\mu(A) < \infty$. By the definition of the outer measure from the pre-measure μ_0, for every $\varepsilon > 0$ there exists intervals $((a_n, b_n])_n$ such that $A \subset \bigcup_n (a_n, b_n]$ and $\sum_n \mu((a_n, b_n]) \leq \mu(A) + \varepsilon$. (Note that even if A is unbounded this is possible with only bounded intervals.) For every n there exists $\delta_n > 0$ such that $F(b_n + \delta_n) < F(b_n) + \varepsilon 2^{-n}$. Thus, taking $U = \bigcup_n (a_n, b_n + \delta_n)$, which is an open set containing A, we get that

$$\begin{aligned}
\mu(U) &\leq \sum_n \mu((a_n, b_n + \delta_n]) = \sum_n F(b_n + \delta_n) - F(a_n) \\
&\leq \sum_n F(b_n) - F(a_n) + \varepsilon = \sum_n \mu((a_n, b_n]) + \varepsilon \leq \mu(A) + 2\varepsilon.
\end{aligned}$$

Taking infimum of the left hand side, and $\varepsilon \to 0$ we have outer regularity.

For inner regularity, first assume that A is bounded. Consider $B = \overline{A} \setminus A$. Fix $\varepsilon > 0$. There exists an open set U such that $B \subset U$ and $\mu(U) < \mu(B) + \varepsilon$. Let $K = \overline{A} \setminus U$. This is a closed bounded set, so it is compact. Also, $K \subset A$ and

$$\mu(B) + \mu(A) = \mu(\overline{A}) \leq \mu(K) + \mu(U) \leq \mu(K) + \mu(B) + \varepsilon,$$

so $\mu(K) \geq \mu(A) - \varepsilon$. Taking supremum of such $K \subset A$ and $\varepsilon \to 0$ we have inner regularity for bounded A.

If A is unbounded: Let $\varepsilon > 0$. For every $r > 0$ there exists a compact $K_r \subset A \cap (-r, r)$ such that $\mu(K_r) \geq \mu(A \cap (-r, r)) - \varepsilon$. Note that $\mu(A \cap (-r, r)) \to \mu(A)$ by monotone convergence. So

$$\sup_{A \supset K \text{ compact}} \mu(K) \geq \lim_{r \to \infty} \mu(A \cap (-r, r)) - \varepsilon = \mu(A) - \varepsilon.$$

Taking $\varepsilon \to 0$ completes the proof. □

7.3. Non-measureable sets

In this section we will show that the Lebesgue measurable sets $\mathcal{L}$ are not all of $2^{\mathbb{R}}$; that is, there exist non-Lebesgue-measurable sets (and specifically non-Borel sets).

► **Exercise 7.5.** Let $Q = \mathbb{Q} \cap [0, 1)$. Define an equivalence relation on $\mathbb{R}$ by $r \sim r'$ if $r - r' \in \mathbb{Q}$. Let R be a set of representatives for the equivalence classes of this relation (chosen by the axiom of choice!).

For every $q \in Q$ let $R_q = \{r + q \pmod 1 \ : \ r \in R \cap [0, 1)\}$.

Show that if $R \in \mathcal{L}$ then also $R_q \in \mathcal{L}$ for all $q \in Q$ and $\lambda(R_q) = \lambda(R)$ where λ is Lebesgue measure.

Show that this leads to a contradiction, so $R \notin \mathcal{L}$.

7.4. Cantor set

The Cantor set $C \subset [0, 1]$ is defined as follows. Start with $C_0 = [0, 1]$. Given C_n define C_{n+1} by "removing the middle third of each interval composing C_n"; formally:

$$C_{n+1} = \tfrac{1}{3} C_n \bigcup \left(\tfrac{2}{3} + \tfrac{1}{3} C_n \right).$$

Then take

$$C = \bigcap_n C_n.$$

► **Exercise 7.6.** Show that C is Lebesgue measurable and has Lebesgue measure 0. (*) Show that C has cardinality of $[0, 1]$.

♣ **Solution to ex:7.6.** :(

Note that for every n the set C_n is the union of two measurable sets so is measurable by induction. Thus, C is measurable as an intersection of measurable sets.

Let us calculate the measure of C_n. Since $C_n \subset [0, 1]$, we have that $\frac{1}{3}C_n \subset [0, \frac{1}{3}]$ and $\frac{2}{3} + \frac{1}{3}C_n \subset [\frac{2}{3}, 1]$. This implies that these sets are disjoint, and

$$\lambda(C_{n+1}) = \lambda(\tfrac{1}{3}C_n) + \lambda(\tfrac{2}{3} + \tfrac{1}{3}C_n) = \tfrac{1}{3}\lambda(C_n) + \tfrac{1}{3}\lambda(C_n) = \tfrac{2}{3}\lambda(C_n).$$

Thus, $\lambda(C_n) = (2/3)^n$ and

$$\lambda(C) \leq \inf_n \lambda(C_n) = 0.$$

:) ✓

Number of exercises in lecture: 6

Total number of exercises until here: 69

Measure Theory

Ariel Yadin

Lecture 8: Functions of measure spaces

8.1. Products

Let $(X, \mathcal{F}), (Y, \mathcal{G})$ be two measurable spaces. What is the natural measurable structure on $X \times Y$? Naturally, any set $A \times B$ for $A \in \mathcal{F}, B \in \mathcal{G}$ should be measurable in the product.

► **Exercise 8.1.** Let $(X, \mathcal{F}), (Y, \mathcal{G})$ be two measurable spaces. Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the natural projections. Show that

$$\sigma(A \times B \ : \ A \in \mathcal{F}, B \in \mathcal{G}) = \sigma(\pi_X^{-1}(A), \pi_Y^{-1}(B) \ : \ A \in \mathcal{F}, B \in \mathcal{G}).$$

♣ **Solution to ex:8.1.** :(

We only need to show that every generator in one σ-algebra is in the other.

If $A \in \mathcal{F}, B \in \mathcal{G}$ then

$$\pi_X^{-1}(A) = A \times Y \qquad \text{and} \qquad \pi_Y^{-1}(B) = X \times Y,$$

which are both in the left σ-algebra.

On the other hand, $A \times B = \pi_X^{-1}(A) \cap \pi_Y^{-1}(B)$ which is in the right σ-algebra. :) ✓

► **Exercise 8.2.** Generalize the previous exercise: If $((X_n, \mathcal{F}_n))_n$ is a sequence of measurable spaces, then

$$\sigma(A_1 \times \cdots \times A_n \times \cdots \ : \ A_n \in \mathcal{F}_n \ \forall \ n) = \sigma(\pi_n^{-1}(A_n) \ : \ A_n \in \mathcal{F}_n),$$

where $\pi_n : X_1 \times \cdots \times X_n \times \cdots \to X_n$ is the natural projection.

✓ We use the notations

$$\bigotimes_n X_n = X_1 \times \cdots \times X_n \times \cdots \qquad \text{and} \qquad \bigotimes_{j=1}^n X_1 \times \cdots \times X_n$$

and

$$\bigotimes_n \mathcal{F}_n = \sigma(\otimes_n A_n \ : \ A_n \in \mathcal{F}_n \ \forall \ n) \qquad \text{and} \qquad \bigotimes_{j=1}^n \mathcal{F}_j = \sigma(\otimes_{j=1}^n A_j \ : \ A_j \in \mathcal{F}_j \ \forall \ j).$$

These are called **product σ-algebras**. Also,

$$\bigotimes_n (X_n, \mathcal{F}_n) = (\otimes_n X_n, \otimes_n \mathcal{F}_n)$$

and similarly for finite products. These latter spaces are called **product (measure) spaces**.

▶ **Exercise 8.3.** Show that $\mathcal{B}(\mathbb{R}^d) = \otimes_{j=1}^d \mathcal{B}(\mathbb{R})$.

▶ **Exercise 8.4.** Show that if $\mathcal{F}_n = \sigma(\mathcal{E}_n)$ for some sets $\mathcal{E}_n$, then

$$\bigotimes_n \mathcal{F}_n = \sigma(\pi_n^{-1}(E_n) \ : \ E_n \in \mathcal{E}_n).$$

♣ **Solution to ex:8.4.** :(

Let

$$\sigma_1 = \bigotimes_n \mathcal{F}_n = \sigma(\pi_n^{-1}(A_n) \ : \ A_n \in \mathcal{F}_n)$$

by a previous exercise. Let $\sigma_2 = \sigma(\pi_n^{-1}(E_n) \ : \ E_n \in \mathcal{E}_n)$.

It is immediate that $\sigma_2 \subset \sigma_1$.

So we only need to show that $\sigma_1 \subset \sigma_2$. For this it suffices to prove that for any n and any $A_n \in \mathcal{F}_n$ we have that $\pi_n^{-1}(A_n) \in \sigma_2$.

Fix n and set $\mathcal{G}_n = \left\{ A \subset X_n \ : \ \pi_n^{-1}(A) \in \sigma_2 \right\}$. Then: $\pi_n^{-1}(\emptyset) = \emptyset \in \sigma_2$ so $\emptyset \in \mathcal{G}_n$. Also, if $A \in \mathcal{G}_n$ then $\pi_n^{-1}(A^c) = \pi_n^{-1}(A)^c \in \sigma_2$, so $A^c \in \mathcal{G}_n$ as well; *i.e.* , $\mathcal{G}$ is closed under complements. If $(A_k)_k \subset \mathcal{G}_n$ is a sequence in $\mathcal{G}_n$ then $\pi_n^{-1}(\bigcup_k A_k) = \bigcup_k \pi_n^{-1}(A_k) \in \sigma_2$; so $\mathcal{G}_n$ is closed under countable unions as well. In conclusion $\mathcal{G}_n$ is a σ-algebra. By definition of σ_2, we have that $\mathcal{E}_n \subset \mathcal{G}_n$, so $\mathcal{F}_n = \sigma(\mathcal{E}_n) \subset \mathcal{G}_n$.

Since this holds for all n, we conclude that for any n and any $A_n \in \mathcal{F}_n$ we have $\pi_n^{-1}(A_n) \in \sigma_2$. :) ✓

Of course now one would like to construct a product measure space from two (or more) measure spaces. For this, it is convenient to first go through the theory of measurable functions.

8.2. Measurable functions

• **Definition 8.1.** Let $(X, \mathcal{F}), (Y, \mathcal{G})$ be measurable spaces. A function $f : X \to Y$ is called $(\mathcal{F}, \mathcal{G})$**-measurable**, or just **measurable**, if $f^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{G}$.

That is, f pulls back measurable sets to measurable sets.

Sometimes we denote this by $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$.

• **Proposition 8.2.** *Let $(X, \mathcal{F}), (Y, \mathcal{G})$ be measurable spaces. Let $f : X \to Y$. Suppose that $\mathcal{G} = \sigma(\mathcal{K})$.*

f is measurable if and only if for every $K \in \mathcal{K}$, $f^{-1}(K) \in \mathcal{F}$.

Proof. One direction is trivial.

For the other direction, verify that $\left\{ A \subset Y \ : \ f^{-1}(A) \in \mathcal{F} \right\}$ is a σ-algebra containing $\mathcal{K}$, and so it contains $\mathcal{G}$ as well. □

► **Exercise 8.5.** Show that if X, Y are topological spaces and $f : X \to Y$ is a continuous function, then f is Borel measurable (*i.e.* $(\mathcal{B}(X), \mathcal{B}(Y))$-measurable).

✓ For a function $f : \mathbb{R}^d \to \mathbb{C}$ we say that f is **Borel** if f is $(\mathcal{B}_d, \mathcal{B}(\mathbb{C}))$-measurable and f is **Lebesgue** if f is $(\mathcal{L}, \mathcal{B}(\mathbb{C}))$-measurable. Note that Borel implies Lebesgue, but the converse is not necessarily true.

► **Exercise 8.6.** Show that for measurable spaces $(X, \mathcal{F}), (Y, \mathcal{G}), (Z, \mathcal{H})$ and measurable functions $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ and $g : (Y, \mathcal{G}) \to (Z, \mathcal{H})$, we have that $g \circ f : X \to Z$ is $(\mathcal{F}, \mathcal{H})$-measurable.

► **Exercise 8.7.** Let $(X, \mathcal{F}), (Y, \mathcal{G}), (Z, \mathcal{H})$ be measurable spaces, and $\pi_X : X \times Y \to X, \pi_Y : X \times Y \to Y$ the natural projections. Show that $f : Z \to X \times Y$ is measurable if and only if $\pi_X \circ f$ and $\pi_Y \circ f$ are measurable.

► **Exercise 8.8.** Let $(X, \mathcal{F}), (Y, \mathcal{G}), (Z, \mathcal{H})$ be measurable spaces. Let $f : X \to Y, g : X \to Z$ be measurable functions. Show that $F : X \to Y \times Z$ defined by $F(x) = (f(x), g(x))$ is a measurable function.

♣ **Solution to ex:8.8.** :(

If $\pi_Y : Y \times Z \to Y, \pi_Z : Y \times Z \to Z$ are the natural projections, then $\pi_Y \circ F = f, \pi_Z \circ F = g$ which are measurable. :) ✓

► **Exercise 8.9.** Show that $f : (X, \mathcal{F}) \to \mathbb{C}$ is measurable if and only if $\mathrm{Re}f, \mathrm{Im}f$ are measurable.

▶ **Exercise 8.10.** Show that if $f, g : (X, \mathcal{F}) \to \mathbb{C}$ are measurable then $f + g$ and fg are also.

♣ **Solution to ex:8.10.** :(

Let $F(x) = (f(x), g(x))$ and $\psi(x, y) = x + y$, $\phi(x, y) = xy$. Then ψ, ϕ are continuous, and F is measurable. So $f + g = \psi \circ F$ and $fg = \phi \circ F$ are measurable. :) ✓

▶ **Exercise 8.11.** [Folland p.48] Show that for $f : (X, \mathcal{F}) \to \mathbb{R}$, if for all $q \in \mathbb{Q}$ we have that $f^{-1}(-\infty, q] \in \mathcal{F}$ then f is measurable.

▶ **Exercise 8.12.** Show that any monotone function $f : \mathbb{R} \to \mathbb{R}$ is Borel.

✓ We will also like to have functions with values $\pm\infty$. One must be careful when adding $\infty - \infty$ and when multiplying $0 \cdot \infty$, but as a topological space we can consider $[-\infty, \infty]$. The Borel σ-algebra is defined in the same way as for $\mathbb{R}$: it is the σ-algebra generated by intervals.

• **Proposition 8.3.** *Let $(f_n)_n$ be a sequence of measurable functions $f_n : (X, \mathcal{F}) \to [-\infty, \infty]$. Then,*

$$\sup_n f_n \qquad \inf_n f_n \qquad \liminf_n f_n \qquad \limsup_n f_n$$

are all measurable.

If the limit $\lim_{n\to\infty} f_n(x) = f(x)$ exists for all x then f is measurable.

Proof. For any $a \le \infty$,

$$(\sup_n f_n)^{-1}(-\infty, a] = \bigcap_n f_n^{-1}(-\infty, a] \in \mathcal{F}.$$

Since $(-\infty, a]$ generate $\mathcal{B}([-\infty, \infty])$ we get that $\sup_n f_n$ is measurable.

Similarly,

$$(\inf_n f_n)^{-1}(-\infty, a] = \bigcup_n f_n^{-1}(-\infty, a] \in \mathcal{F}.$$

A bit more cumbersome to check, but not too difficult is

$$(\liminf f_n)^{-1}(-\infty, a] = \bigcap_n \bigcup_{k \ge n} f_k^{-1}(-\infty, a],$$

and

$$(\limsup f_n)^{-1}(-\infty, a] = \bigcup_n \bigcap_{k \ge n} f_k^{-1}(-\infty, a].$$

Which gives the measurability of $\liminf f_n$ and $\limsup f_n$.

Finally, if $f = \lim f_n$ exists then $f = \limsup f_n = \liminf f_n$ and so is measurable. □

► **Exercise 8.13.** Show that for $f : (X, \mathcal{F}) \to [-\infty, \infty]$,

$$f^+ = \max\{0, f\} \qquad \text{and} \qquad f^- = \max\{0, -f\}$$

are measurable if f is.

► **Exercise 8.14.** Show that for measurable $f : (X, \mathcal{F}) \to \mathbb{C}$ the functions $|f| : X \to \mathbb{R}$ and $\mathrm{sgn} f : X \to \mathbb{R}$ defined by $|f|(x) = |f(x)|$ and

$$\mathrm{sgn} f(x) = \begin{cases} \frac{f(x)}{|f(x)|} & f(x) \ne 0, \\ 0 & f(x) = 0, \end{cases}$$

are measurable.

Show that $\arg f$ is measurable.

▶ **Exercise 8.15.** [Folland p.48] Let $(X, \mathcal{F})$ be a measurable space. Let $f : X \to [-\infty, \infty]$ and let $Y = f^{-1}(\mathbb{R})$. Show that f is measurable if and only if $f^{-1}(\infty) \in \mathcal{F}, f^{-1}(-\infty) \in \mathcal{F}$ and $f\big|_Y$ is measurable as a function $f\big|_Y : (Y, \mathcal{F}_Y) \to \mathbb{R}$, where $\mathcal{F}_Y = \{A \cap Y \ : \ A \in \mathcal{F}\}$.

▶ **Exercise 8.16.** [Folland p.48] Let $(X, \mathcal{F})$ be a measurable space. Let $f, g : X \to [-\infty, \infty]$.

Define $fg(x) = f(x)g(x)$ where $\pm\infty \cdot 0 = 0 \cdot (\pm\infty) = 0$ and $x \cdot (\pm\infty) = \pm\infty \cdot x = \pm\infty$ for $x > 0$ and $x \cdot (\pm\infty) = \pm\infty \cdot x = \mp\infty$ if $x < 0$.

Show that if f, g are measurable then fg is measurable.

For $a \in [-\infty, \infty]$ define

$$(f+g)_a(x) = \begin{cases} f(x) + g(x) & \text{if } f(x) \neq -g(x) = \pm\infty \text{ or } |f(x)| \wedge |g(x)| < \infty, \\ a & \text{if } f(x) = -g(x) = \pm\infty \end{cases}$$

(that is, replacing the problematic $\infty - \infty$ with a).

Show that if f, g are measurable then $(f+g)_a$ is measurable.

▶ **Exercise 8.17.** Show that if $(f_n)_n$ is a sequence of measurable functions on $(X, \mathcal{F})$ then $A = \{x \ : \ \lim_{n\to\infty} f_n(x) \text{ exists } \}$ is a measurable set. Show that for any $a \in \mathbb{R}$,

$$f(x) = \begin{cases} \lim_{n\to\infty} f_n(x) & \text{if the limit exists,} \\ a & \text{otherwise} \end{cases}$$

is measurable.

8.3. Simple functions

• **Definition 8.4** (Simple functions)**.** Let $(X, \mathcal{F})$ be a measurable space.

The **indicator function** of a set $A \subset X$, denoted $\mathbf{1}_A$ is the function

$$\mathbf{1}_A : X \to \mathbb{R} \qquad \mathbf{1}_A(x) := \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

A **simple function** is a *measurable* function $f : X \to [0, \infty)$ such that $f(X)$ is a finite set.

▶ **Exercise 8.18.** Let $(X, \mathcal{F})$ be a measurable space, and let $A \subset X$. Show that $\mathbf{1}_A$ is measurable if and only if $A \in \mathcal{F}$.

▶ **Exercise 8.19.** Let $(X, \mathcal{F})$ be a measurable space. Show that for any simple function f we can write $f = \sum_{j=1}^{n} a_j \mathbf{1}_{A_j}$ where $0 < a_1 < a_2 < \cdots < a_n$ and $A_j \in \mathcal{F}$ for all j. Show that in fact, we have the **standard representation**

$$f = \sum_{0 < a \in f(X)} a \mathbf{1}_{f^{-1}(a)}.$$

♣ **Solution to ex:8.19.** :(

Since $f(X)$ is finite, and since $f \geq 0$, we may write $f(X) = \{a_1 < a_2 < \cdots < a_n\}$ for $a_1 > 0$ (if $0 \notin f(X)$) or $f(X) = \{0 = a_0 < a_1 < \cdots < a_n\}$ (in the case that $0 \in f(X)$).

Since f is measurable, $A_j := f^{-1}(a_j) \in \mathcal{F}$ for all j. Finally, since $X = f^{-1}(f(X)) = \biguplus_j f^{-1}(a_j) = \biguplus_j A_j$, for every $x \in X$ there exists a unique $j(x)$ such that $x \in A_{j(x)}$.

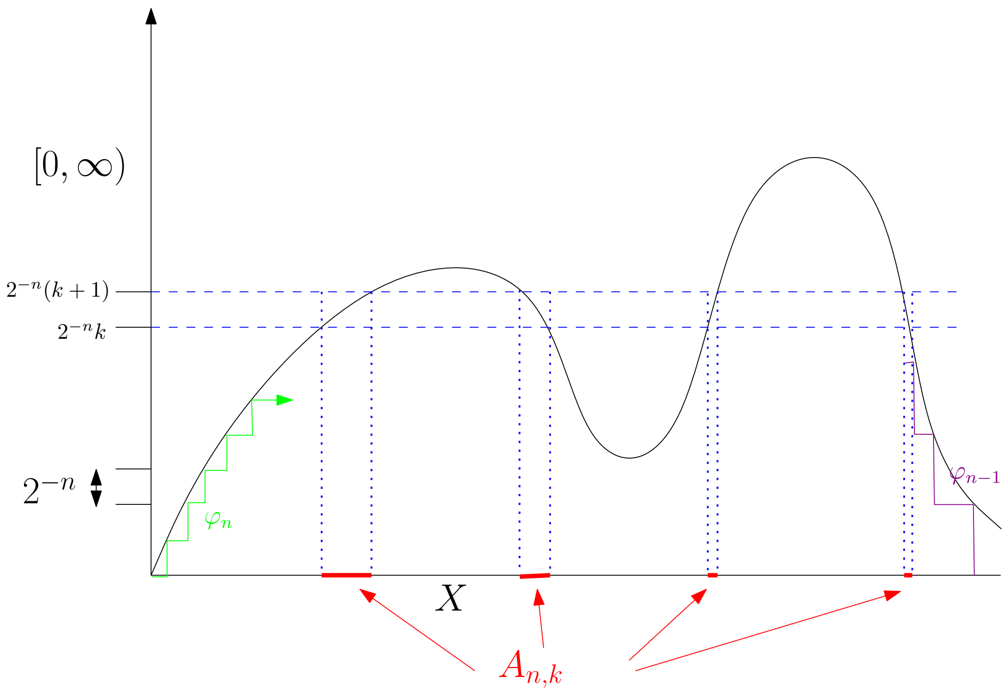

FIGURE 3. Approximating functions with simple functions

Note that $f(x) = a_{j(x)} = a_{j(x)}\mathbf{1}_{A_{j(x)}}(x)$. Since $(A_j)_j$ are disjoint, if $\mathbf{1}_{A_j}(x) \neq 0$ then $\mathbf{1}_{A_i}(x) = 0$ for all $i \neq j$. Altogether this gives

$$f(x) = \sum_{a \in f(X)} a\mathbf{1}_{\{f(a)=x\}} = \sum_{0<a\in f(X)} a\mathbf{1}_{f^{-1}(a)}(x).$$

:) ✓

The following is a very useful building block in the theory of measurable functions.

• **Proposition 8.5.** *Let $(X, \mathcal{F})$ be a measurable space. Let $f : X \to \mathbb{C}$ be a Borel function.*

- *If $f \geq 0$ then there exists a monotone sequence of simple functions $0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq \varphi_n \leq \cdots$ such that $\varphi_n \nearrow f$. In fact, the convergence is uniform on any set for which f is bounded.*
- *If f is real-valued then $f = f^+ - f^-$ and f^+, f^- are non-negative Borel functions.*
- *In general, $f = \mathrm{Re}f + i\mathrm{Im}f$ where $\mathrm{Re}f, \mathrm{Im}f$ are real valued and Borel. So $f = f_1 - f_2 + i(f_3 - f_4)$ where all f_j are non-negative Borel functions.*

Proof. We only prove the first bullet. The other bullets have actually been proven in previous exercises.

If $f \geq 0$ then for every n define $\varphi_n := \min\{n, 2^{-n}\lfloor 2^n f \rfloor\}$.

Note that $\varphi_n(X) \subset \{2^{-n}k \ : \ k \in \mathbb{N} \ , \ k \leq 2^n n\}$ which is finite, so φ_n is simple (why is it measurable?). Also, one may verify that for the sets $A_{n,k} = f^{-1}[2^{-n}k, 2^{-n}(k+1))$ and $E_n = f^{-1}[n + 2^{-n}, \infty)$ we have that the standard representation of φ_n is

$$\varphi_n = \sum_{k=0}^{2^n n} 2^{-n} k \mathbf{1}_{A_{n,k}} + n \mathbf{1}_{E_n}.$$

To show that $\varphi_n \leq \varphi_{n+1}$, note that $2\lfloor r \rfloor \leq \lfloor 2r \rfloor$ for any $r \geq 0$, so

$$\varphi_n = n \wedge 2^{-n}\lfloor 2^n f \rfloor \leq (n+1) \wedge 2^{-n}\lfloor 2^n f \rfloor \leq (n+1) \wedge 2^{-n}\tfrac{1}{2}\lfloor 2 \cdot 2^n f \rfloor = \varphi_{n+1}.$$

Now, if A is a set on which f is bounded, say $\sup_{x \in A} f(x) = M < \infty$, then for all $n > M$ we have that $\sup_{x \in A} 2^{-n}\lfloor 2^n f(x) \rfloor < n$ so $\varphi_n(x) = 2^{-n}\lfloor 2^n f(x) \rfloor$ for all $x \in A$. Thus, $\sup_{x \in A} |\varphi_n(x) - f(x)| \leq 2^{-n}$ for all $n > M$. So $\varphi_n \to f$ uniformly on A. □

▶ **Exercise 8.20.** Let $f : (X, \mathcal{F}) \to [0, \infty]$ be a measurable function. Show that there exists a sequence of simple functions $0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq \varphi_n \leq \cdots$ such that $\varphi_n \nearrow f$.

♣ **Solution to ex:8.20.** :(

Write $f = f\mathbf{1}_A + \infty\mathbf{1}_{A^c}$ where $0 \cdot \infty = \infty \cdot 0 = 0$ and $A^c = f^{-1}(\infty)$.

$f\mathbf{1}_A$ is measurable and non-negative, so there is a monotone sequence a simple functions $\psi_n \nearrow f\mathbf{1}_A$. Define $\varphi_n = \psi_n \mathbf{1}_A + n\mathbf{1}_{A^c}$. :) ✓

Number of exercises in lecture: 20

Total number of exercises until here: 89

Measure Theory

Ariel Yadin

Lecture 9: Integration: positive functions

9.1. Integration of simple functions

Let $(X, \mathcal{F}, \mu)$ be a measure space. Let φ be a simple function, with standard representation $\varphi = \sum_{j=1}^n a_j \mathbf{1}_{A_j}$. Define the integral of φ with respect to μ as

$$\int \varphi d\mu := \sum_{j=1}^n a_j \mu(A_j).$$

For $A \in \mathcal{F}$ define

$$\int_A \varphi d\mu := \int \varphi \mathbf{1}_A d\mu.$$

• **Proposition 9.1.** *Let φ, ψ be simple functions on a measure space $(X, \mathcal{F}, \mu)$. Then:*

- *For all $r > 0$ we have $\int r\varphi d\mu = r \int \varphi d\mu$.*
- *If $\varphi = \sum_{j=1}^n a_j \mathbf{1}_{A_j}$ for pairwise disjoint $(A_j)_{j=1}^n$, then $\int \varphi d\mu = \sum_{j=1}^n a_j \mu(A_j)$.*
- *$\int (\varphi + \psi) d\mu = \int \varphi d\mu + \int \psi d\mu$.*
- *If $\varphi \leq \psi$ then $\int \varphi d\mu \leq \int \psi d\mu$.*

Proof. For $r > 0$, $r\varphi$ is a simple function such that $r\varphi(X) = \{ra \ : \ a \in \varphi(X)\}$. So,

$$r\varphi = \sum_{0 < a \in \varphi(X)} ra \mathbf{1}_{\varphi^{-1}(a)}$$

is the standard representation of $r\varphi$. Thus by definition,

$$\int r\varphi d\mu = \sum_{0 < a \in \varphi(X)} ra\mu(\varphi^{-1}(a)) = r \int \varphi d\mu.$$

Also, $X = \biguplus_{a \in \varphi(X)} \varphi^{-1}(a)$. Thus, if $\varphi = \sum_{j=1}^n a_j \mathbf{1}_{A_j}$ then $A_j = \biguplus_{a \in \varphi(X)} A_j \cap \varphi^{-1}(a)$. Note that for $x \in A_j \cap \varphi^{-1}(a)$ we have that $a = \varphi(x) = a_j$. Thus, $a_j \mu(A_j \cap \varphi^{-1}(a)) = a\mu(A_j \cap \varphi^{-1}(a))$. Also, for $B = \left(\biguplus_{j=1}^n A_j \right)^c$ we have that $a\mu(B \cap \varphi^{-1}(a)) = 0$ because either $B \cap \varphi^{-1}(a) = \emptyset$ or for $x \in B \cap \varphi^{-1}(a)$, $a = \varphi(x) = \sum_{j=1}^n a_j \mathbf{1}_{A_j}(x) = 0$. Since $X = \biguplus_{j=1}^n A_j \biguplus B$,

$$\sum_{j=1}^n a_j \mu(A_j) = \sum_{j=1}^n \sum_{a \in \varphi(X)} a\mu(A_j \cap \varphi^{-1}(a)) = \sum_{a \in \varphi(X)} a\mu(\varphi^{-1}(a)) = \int \varphi d\mu.$$

Now, $(\varphi + \psi)(X) = \{a + b \ : \ a \in \varphi(X), b \in \psi(X)\}$ so it is indeed a simple function. Note that for $E_{a,b} = \varphi^{-1}(a) \cap \psi^{-1}(b)$, these sets are pairwise disjoint, and

$$\varphi + \psi = \sum_{a \in \varphi(X), b \in \psi(X)} (a + b)\mathbf{1}_{E_{a,b}}.$$

Similarly,

$$\varphi = \sum_{a \in \varphi(X), b \in \psi(X)} a\mathbf{1}_{E_{a,b}} \qquad \text{and} \qquad \psi = \sum_{a \in \varphi(X), b \in \psi(X)} b\mathbf{1}_{E_{a,b}}.$$

So by the above,

$$\int (\varphi + \psi) d\mu = \sum_{a \in \varphi(X), b \in \psi(X)} (a + b)\mu(E_{a,b}) = \int \varphi d\mu + \int \psi d\mu.$$

Finally, if $\varphi \leq \psi$ we claim that for all $a \in \varphi(X), b \in \psi(X)$ we have that $a\mu(E_{a,b}) \leq b\mu(E_{a,b})$, because either $E_{a,b} = \emptyset$ or if $x \in E_{a,b}$ then $a = \varphi(x) \leq \psi(x) = b$. Thus,

$$\int \varphi d\mu = \sum_{a \in \varphi(X), b \in \psi(X)} a\mu(E_{a,b}) \leq \sum_{a \in \varphi(X), b \in \psi(X)} b\mu(E_{a,b}) = \int \psi d\mu.$$

□

• **Proposition 9.2.** *Let φ be a simple functions on a measure space $(X, \mathcal{F}, \mu)$. The map $A \mapsto \int_A \varphi d\mu$ is a measure on $(X, \mathcal{F})$.*

Proof. The function $\varphi \mathbf{1}_\emptyset$ is just the zero function, which has 0 integral by definition.

For any $E \in \mathcal{F}$,

$$\varphi \mathbf{1}_E = \sum_{a \in \varphi(X)} a\mathbf{1}_{\varphi^{-1}(a) \cap E}.$$

The sets $(\varphi^{-1}(a) \cap E)_{a \in \varphi(X)}$ are pairwise disjoint so

$$\int_E \varphi d\mu = \sum_{a \in \varphi(X)} a\mu(\varphi^{-1}(a) \cap E).$$

Now if $A = \biguplus_n A_n$ then

$$\int_A \varphi d\mu = \sum_{a \in \varphi(X)} a\mu(\varphi^{-1}(a) \cap A) = \sum_{a \in \varphi(X)} a \sum_n \mu(\varphi^{-1}(a) \cap A_n) = \sum_n \int_{A_n} \varphi d\mu.$$

□

9.2. Integration of positive functions

We use $L^+(X, \mathcal{F})$ to denote the set of measurable functions $f : (X, \mathcal{F}) \to [0, \infty]$.

• **Definition 9.3.** Let $(X, \mathcal{F}, \mu)$ be a measure space. For $f \in L^+(X, \mathcal{F})$ define

$$\int f d\mu := \sup \Big\{ \int \varphi d\mu \ : \ 0 \leq \varphi \leq f \ , \ \varphi \text{ is simple } \Big\}.$$

► **Exercise 9.1.** Show that if $f \leq g$ for $f, g \in L^+(X, \mathcal{F})$ then $\int f d\mu \leq \int g d\mu$. Show that for any $c > 0$, $\int cf d\mu = c \int f d\mu$.

♣ **Solution to ex:9.1.** :(

This simple functions participating in the supremum for f also participate in the supremum for g.

If $c > 0$ then for any simple function φ, $\varphi \leq f$ if and only if $c\varphi \leq cf$. Taking supremums over $\int c\varphi d\mu = c \int \varphi d\mu$ yields the result. :) ✓

It is usually difficult to compute supremums over such big sets (all simple functions dominated by some $f \in L^+$). The next fundamental result will make life much simpler when wishing to compute integrals of positive functions.

⋆⋆⋆ **THEOREM 9.4** (Monotone Convergence)**.** *Let $(X, \mathcal{F}, \mu)$ be a measure space. If $(f_n)_n$ is a sequence in $L^+(X, \mathcal{F})$ such that $0 \leq f_n \leq f_{n+1}$ for all n (*i.e.* a monotone sequence), then for the limit $f(x) = \lim_{n\to\infty} f_n(x)$ (which always exists as $f = \sup_n f_n$),*

$$\int f d\mu = \lim_{n\to\infty} f_n d\mu.$$

Proof. Because $f_n \leq f$ for all n we have that $\int f_n \leq \int f$ and so $\lim \int f_n \leq \int f$. We are left with proving the other inequality.

Let $0 < \varepsilon < 1$ and let φ be a simple function such that $0 \leq \varphi \leq f$. Define

$$A_n = \{x \in X \ : \ f_n(x) \geq (1-\varepsilon)\varphi(x)\} = \{f_n \geq (1-\varepsilon)\varphi\} \, .$$

Note that $A_n \subset A_{n+1}$ because $f_n \leq f_{n+1}$. Thus $(A_n)_n$ is an increasing sequence. Since $A \mapsto \int_A \varphi d\mu$ is a measure on $(X, \mathcal{F})$, we get that

$$\int_{A_n} \varphi d\mu \to \int \varphi d\mu$$

because

$$\begin{aligned}\bigcup_n A_n &= \{x \ : \ \exists \ n \ , \ f_n(x) \geq (1-\varepsilon)\varphi(x)\} = \left\{x \ : \ \sup_n f_n(x) \geq (1-\varepsilon)\varphi(x)\right\} \\ &= \{x \ : \ f(x) \geq (1-\varepsilon)\varphi(x)\} = X.\end{aligned}$$

Since $f_n \geq f_n \mathbf{1}_{A_n} \geq (1-\varepsilon)\varphi \mathbf{1}_{A_n}$, we get that

$$\int f_n d\mu \geq (1-\varepsilon) \cdot \int_{A_n} \varphi d\mu \to (1-\varepsilon) \cdot \int \varphi d\mu.$$

Taking supremum over φ and $\varepsilon \to 0$ we get that

$$\lim_{n\to\infty} \int f_n d\mu \geq \int f d\mu,$$

which completes the proof. □

• **Proposition 9.5** (Poor man's Fubini). *If $(f_n)_n$ is a sequence of functions in $L^+(X, \mathcal{F})$ then*

$$\int \sum_n f_n d\mu = \sum_n \int f_n d\mu.$$

Proof. If $f, g \in L^+(X, \mathcal{F})$, then let $(\varphi_n)_n, (\psi_n)_n$ be monotone sequences of simple functions such that $\varphi_n \nearrow f$ and $\psi_n \nearrow g$. Then, $\varphi_n + \psi_n \nearrow f + g$. So, using the Monotone Convergence Theorem,

$$\int (f+g) d\mu = \lim_{n\to\infty} \int (\varphi_n + \psi_n) d\mu = \int f d\mu + \int g d\mu.$$

Now, for a sequence $(f_n)_n$ in $L^+(X,\mathcal{F})$, for any n,

$$\int \sum_{k=1}^n f_k d\mu = \sum_{k=1}^n \int f_k d\mu.$$

Since $(f_n)_n$ are all non-negative, $\sum_{k=1}^n f_k \nearrow \sum_n f_n$, so by Monotone Convergence again,

$$\sum_n \int f_n d\mu = \lim_{n\to\infty} \int \sum_{k=1}^n f_k d\mu = \int \sum_n f_n d\mu.$$

□

✓ For a measure μ we say that a measurable set A occurs *almost everywhere*, or a.e., if $\mu(A^c)=0$. sometimes we stress the dependence on the measure by saying μ-a.e.

✓ For a function f we may sometimes shorthand $\{f \leq a\} = \{x \ : \ f(x) \leq a\}$, and similarly for other measurable sets.

• **Proposition 9.6.** *For $f \in L^+(X,\mathcal{F})$, $f=0$ a.e. (that is, $\mu(\{x \ : \ f(x) \neq 0\})=0$) if and only if $\int f d\mu = 0$.*

Proof. If f is a simple function then $\int f d\mu = \sum_{0<a\in f(X)} a\mu(f^{-1}(a))$ and this is 0 if and only if for every $a \in f(X)$, $a\mu(f^{-1}(a))=0$, which is if and only if $\mu(\{x \ : \ f(x)>0\}) = 0$.

Now, if $f \in L^+$, then let $\varphi_n \nearrow f$ be an approximating monotone sequence of simple functions. Since $0 \leq \varphi_n \leq f$ we have that $f=0$ implies that $\varphi_n = 0$ for all n and so $\int f = \lim \int \varphi_n = 0$.

In the other direction, if since $\{f>0\} = \bigcup_n \left\{f > \frac{1}{n}\right\}$ we have that if $\mu(f>0)>0$ then there exists n such that $\mu(f>\frac{1}{n})>0$. For $A = \left\{f > \frac{1}{n}\right\}$,

$$\int_A f d\mu \geq \tfrac{1}{n} \int \mathbf{1}_A d\mu = \tfrac{1}{n}\mu(A) > 0.$$

□

► **Exercise 9.2.** Show that if $(f_n)_n$ is a monotone sequence in L^+ such that $f_n \nearrow f$ a.e., then $\int f_n d\mu \to \int f d\mu$.

♣ **Solution to ex:9.2.** :(

Let $A = \{f_n \nearrow f\}$. So $g_n = f_n \mathbf{1}_A$ is a monotone sequence on L^+ that increases to $g_n \nearrow g := f\mathbf{1}_A$. Also, $f_n(1 - \mathbf{1}_A) = f_n \mathbf{1}_{A^c}$, $f(1 - \mathbf{1}_A) = f\mathbf{1}_{A^c} \in L^+$ and $\mu(A^c) = 0$, so

$$\int f_n d\mu = \int g_n d\mu \qquad \text{and} \qquad \int f d\mu = \int g d\mu.$$

Thus,

$$\int f d\mu = \int g d\mu = \lim_{n\to\infty} \int g_n d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

:) ✓

• **Lemma 9.7** (Fatou's Lemma). *Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $(f_n)_n$ be a sequence on $L^+(X, \mathcal{F})$. Then,*

$$\int \liminf f_n d\mu \leq \liminf_{n\to\infty} \int f_n d\mu.$$

Proof. For every $j \geq k$ we have that $\inf_{n\geq k} f_n \leq f_j$ and so $\int \inf_{n\geq k} f_n d\mu \leq \inf_{j\geq k} \int f_j d\mu$.

The sequence $g_k := \inf_{n\geq k} f_n$ is a monotone sequence in L^+ converging to $g_k \nearrow \liminf f_n$. By Monotone Convergence,

$$\int \liminf f_n d\mu = \lim_{k\to\infty} \int g_k d\mu \leq \lim_{k\to\infty} \inf_{j\geq k} \int f_j d\mu = \liminf_{n\to\infty} \int f_n d\mu.$$

□

► **Exercise 9.3.** Assume that $(f_n)_n$ is a sequence in L^+ such that $\int \sup f_n d\mu < \infty$. Show that

$$\limsup_{n\to\infty} \int f_n d\mu \leq \int \limsup f_n d\mu.$$

♣ **Solution to ex:9.3.** :(

Define $g_n = (\sup f_n) - f_n$. Then $(g_n)_n \subset L^+$. Since g_n, f_n are non-negative functions,

$\int f_n \leq \int g_n + \int f_n = \int \sup f_n < \infty$, and we can subtract $\int f_n$ from both sides to get that $\int g_n = \int \sup f_n - \int f_n$ for all n. Also, since $\liminf g_n, \limsup f_n$ are non-negative functions, and since $\liminf g_n + \limsup f_n = \sup f_n$, we have that

$$\int \limsup f_n d\mu \leq \int \liminf g_n d\mu + \int \limsup f_n d\mu = \int \sup f_n d\mu < \infty,$$

and subtracting $\int \limsup f_n d\mu$ from both sides we have that

$$\int \liminf g_n d\mu = \int \sup f_n d\mu - \int \limsup f_n d\mu.$$

However, by Fatou's Lemma,

$$\int \liminf g_n d\mu \leq \liminf \int g_n d\mu = \int \sup f_n d\mu - \limsup \int f_n d\mu.$$

Since $\int \sup f_n d\mu < \infty$ we may subtract it from both sides. :) ✓

► **Exercise 9.4.** Let $f \in L^+(X, \mathcal{F})$ for a measure space $(X, \mathcal{F}, \mu)$. Show that $\nu(A) := \int_A f d\mu$ is a measure on $(X, \mathcal{F})$. Show that for any $g \in L^+(X, \mathcal{F})$,

$$\int g d\nu = \int g f d\mu.$$

♣ **Solution to ex:9.4.** :(

$\nu(\emptyset) = 0$ is simple.

If $A = \biguplus_n A_n$ then $f\mathbf{1}_A = \sum_n f\mathbf{1}_{A_n}$ and so

$$\nu(A) = \int_A f d\mu = \sum_n \int_{A_n} f d\mu = \sum_n \nu(A_n)$$

(by poor man's Fubini).

This shows that ν is a measure.

Now, if g is a simple function with standard representation $g = \sum_{j=1}^n a_j \mathbf{1}_{A_j}$, then

$$\int g d\nu = \sum_{j=1}^n a_j \nu(A_j) = \sum_{j=1}^n a_j \int_{A_j} f d\mu = \int g f d\mu.$$

Now for general $g \in L^+$, let $\varphi_n \nearrow g$ be a monotone sequence of simple functions approximating g. So $\varphi_n f \nearrow gf$. Monotone Convergence guaranties that

$$\int g d\nu = \lim_{n\to\infty} \int \varphi_n d\nu = \lim_{n\to\infty} \int \varphi_n f d\mu = \int g f d\mu.$$

:) ✓

▶ **Exercise 9.5.** [Folland, p.52, ex.13] Suppose $(f_n)_n$ is a sequence of non-negative functions such that $f_n \to f$ a.e. and $\int f_n \to \int f < \infty$.

Show that for any measurable A, $\int_A f_n \to \int_A f$.

Show that this is not necessarily true if $\int f = \infty$.

♣ **Solution to ex:9.5.** :(

Since $f_n \mathbf{1}_A \to f \mathbf{1}_A$, by Fatou's Lemma, for any A,

$$\int_A f \leq \liminf \int_A f_n.$$

Since $\int f < \infty$,

$$\int_A f = \int f - \int_{A^c} f \geq \int f - \liminf \int_{A^c} f_n = \limsup(\int f_n - \int_{A^c} f_n) = \limsup \int_A f_n.$$

Hence,

$$\int_A f \leq \liminf \int_A f_n \leq \limsup \int_A f_n \leq \int_A f.$$

Now for a counter-example when $\int f = \infty$. Set

$$f_n = \mathbf{1}_{[0,n^{-1}]} n^2 + \mathbf{1}_{[n^{-1},n)}.$$

So $f_n \to \mathbf{1}_{[0,\infty)} =: f$. Also,

$$\int f_n = n + n - n^{-1} \to \infty = \int f.$$

However, for $A = [0,1]$,

$$\int_A f_n = n + 1 - n^{-1} \to \infty \neq 1 = \int_A f.$$

:) ✓

► **Exercise 9.6.** Let $(X, \mathcal{F}, \mu)$ be a measure space. Assume that if $(f_n)_n$ is a sequence in $L^+(X, \mathcal{F})$ such that $f_{n+1} \leq f_n$ for all n. Assume that $\int f_1 d\mu < \infty$ and that $f_n \searrow f$ a.e.

Show that $\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$.

► **Exercise 9.7.** Let $(X, \mathcal{F}, \mu)$ be a measure space and $f \in L^+(X, \mathcal{F})$ such that $\int f d\mu < \infty$.

Show that $\{f = \infty\} = f^{-1}(\infty)$ is a null set.

Show that $\{f > 0\}$ is σ-finite (i.e. can be written as a countable union of finite-measure sets).

♣ **Solution to ex:9.7.** :(

If $\mu(f = \infty) > 0$ then

$$\infty = \infty \cdot \mu(f = \infty) \leq \int_{\{f=\infty\}} f d\mu \leq \int f d\mu < \infty,$$

a contradiction.

Write $\{f > 0\} = \{f = \infty\} \uplus \biguplus_{n=1}^{\infty} \{n - 1 < f \leq n\}$. We have already seen that $\{f = \infty\}$ is a null set (and specifically has finite measure). For any $n \geq 1$, if $\mu(n - 1 < f \leq n) = \infty$ then

$$\infty = (n-1) \cdot \mu(n - 1 < f \leq n) \leq \int_{\{n-1<f\leq n\}} f d\mu \leq \int f d\mu < \infty,$$

a contradiction. So $\mu(n - 1 < f \leq n) < \infty$ for all n. :) ✓

Number of exercises in lecture: 7

Total number of exercises until here: 96

Measure Theory

Ariel Yadin

Lecture 10: Integration: general functions

We have already seen that any measurable function can be decomposed into real and imaginary parts, and those into positive and negative parts each. So essentially we can split the task of integrating a functions into integrating four positive functions. The only thing we have to be careful about is $\infty - \infty$.

10.1. Real valued functions

• **Definition 10.1.** Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $f : (X, \mathcal{F}) \to \mathbb{R}$ be a measurable function.

Consider the integrals $I^+ = \int f^+ d\mu, I^- = \int f^- d\mu$ ($f^+ = 0 \vee f, f^- = 0 \vee -f$).

If at least one integral I^+, I^- is finite then we define

$$\int f d\mu := \int f^+ d\mu - \int f^- d\mu$$

and we say the integral $\int f d\mu$ exists. If $I^+ = I^- = \infty$ then $\int f d\mu$ does not exist (or is undefined).

If $I^+ < \infty$ and $I^- < \infty$ then we say that f is **integrable**.

As usual we define $\int_A f d\mu = \int f \mathbf{1}_A d\mu$.

► **Exercise 10.1.** Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $f : (X, \mathcal{F}) \to \mathbb{R}$ be a measurable function. Show that f is integrable if and only if $\int |f| d\mu < \infty$.

• **Proposition 10.2.** *For a measure space $(X, \mathcal{F}, \mu)$ the set of integrable real-valued functions is a vector space, and the integral is a linear functional on this space.*

Proof. If f, g are integrable then

$$\int |f+\alpha g|d\mu \leq \int (|f|+|\alpha||g|)d\mu \leq \int |f|d\mu + |\alpha|\int |g|d\mu < \infty.$$

So $f+\alpha g$ is also integrable.

Now, write $h = f+g$. So $h^+ - h^- = f^+ - f^- + g^+ - g^-$, which implies that $h^+ + f^- + g^- = h^- + f^+ + g^+$. Since these are all positive functions,

$$\int h^+ + \int f^- + \int g^- = \int h^- + \int f^+ + \int g^+,$$

which after rearranging gives

$$\int h = \int h^+ - \int h^- = \int f^+ - \int f^- + \int g^+ - \int g^- = \int f + \int g.$$

If $\alpha > 0$ then $(\alpha f)^+ = \alpha f^+, (\alpha f)^- = \alpha f^-$ and $(-\alpha f)^+ = \alpha f^-, (-\alpha f)^- = \alpha f^+$. So

$$\int \alpha f = \int \alpha f^+ - \int \alpha f^- = \alpha \int f \qquad \text{and} \qquad \int (-\alpha f) = \int \alpha f^- - \int \alpha f^+ = -\alpha \int f.$$

□

10.2. Complex valued functions

• **Definition 10.3.** Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f : (X, \mathcal{F}) \to \mathbb{C}$ be a measurable function.

We say that f is **integrable** if $\int |f|d\mu < \infty$ (this makes sense because $|f| \in L^+(X, \mathcal{F})$).

We define

$$\int f d\mu = \int \text{Re} f d\mu + i \int \text{Im} f d\mu.$$

As usual we define $\int_A f d\mu = \int f \mathbf{1}_A d\mu$.

The set of all complex valued integrable functions is denoted $L^1(X, \mathcal{F}, \mu) = L^1(X) = L^1(\mu)$.

► **Exercise 10.2.** $f \in L^1(X, \mathcal{F}, \mu)$ if and only if $\text{Re} f, \text{Im} f$ are both integrable.

Thus, the integral of a complex function is well defined. It is a (complex) linear functional the vector space $L^1(X, \mathcal{F}, \mu)$.

♣ **Solution to ex:10.2.** :(

The first assertion follows since $|f| \leq |\text{Re}f| + |\text{Im}f| \leq 2|f|$. :) ✓

▶ **Exercise 10.3.** Show that for $f \in L^1$,

$$\left|\int f d\mu\right| \leq \int |f| d\mu.$$

♣ **Solution to ex:10.3.** :(

If $\int f = 0$ this is immediate.

Otherwise, if f is real valued then $|f| = f^+ + f^-$, so

$$\left|\int f\right| = \left|\int f^+ - \int f^-\right| \leq \int f^+ + \int f^- = \int |f|.$$

If f is complex valued and $\int f \neq 0$, then set $\alpha = \frac{|\int f|}{\int f} \in \mathbb{C}$. So $\alpha \int f = \int \alpha f \in \mathbb{R}$ and

$$\left|\int f\right| = \int \alpha f = \text{Re} \int \alpha f = \int \text{Re}(\alpha f) \leq \int |\text{Re}(\alpha f)| \leq \int |\alpha||f| = \int |f|.$$

:) ✓

▶ **Exercise 10.4.** Show that for $f \in L^1$ the set $\{f \neq 0\}$ is σ-finite.

• **Proposition 10.4.** *For $f, g \in L^1(X, \mathcal{F}, \mu)$ we have that the following are equivalent.*

- $f = g$ *a.e.*
- $\int |f - g| d\mu = 0$.
- *For every* $A \in \mathcal{F}$, $\int_A f d\mu = \int_A g d\mu$.

Proof. If $f = g$ a.e. then $|f - g| = 0$ a.e. so $\int |f - g| d\mu = 0$.

If $\int |f-g|d\mu = 0$ then for any $A \in \mathcal{F}$,

$$|\int_A f d\mu - \int_A g d\mu| \leq \int \mathbf{1}_A |f-g| d\mu \leq \int |f-g| d\mu = 0.$$

If $\int_A f d\mu = \int_A g d\mu$ for all $A \in \mathcal{F}$ then also $\int_A \mathrm{Re}(f-g) d\mu = \int_A \mathrm{Im}(f-g) d\mu = 0$ for all $A \in \mathcal{F}$. So without loss of generality we may assume that f, g are real valued. Set $A = \{f > g\}$ and $B = \{f < g\}$. We then have

$$\int |f-g| d\mu = \int_A (f-g) d\mu + \int_B (g-f) d\mu = 0.$$

So $|f-g| = 0$ a.e., and $f = g$ a.e. □

10.3. Convergence

► **Exercise 10.5.** Let $(f_n)_n$ be a sequence in $L^1(X, \mathcal{F}, \mu)$ such that $f_n \to f$ uniformly (*i.e.* for every $\varepsilon > 0$ there exists n_0 such that for all $n > n_0$, $\sup_x |f_n(x) - f(x)| < \varepsilon$).

Show that if $\mu(X) < \infty$ then $\int f_n d\mu \to \int f d\mu$.

Show that if $\mu(X) = \infty$ this is not necessarily true.

★★★ **THEOREM 10.5** (Dominated Convergence Theorem)**.** *Let $(f_n)_n$ be a sequence in $L^1(X, \mathcal{F}, \mu)$ such that $f_n \to f$ a.e. and there exists $g \in L^1(X, \mathcal{F}, \mu)$ such that $|f_n| \leq g$ for all n.*

Then $f \in L^1(X, \mathcal{F}, \mu)$ and $\int f_n d\mu \to \int f d\mu$.

Proof. Let $E = \{f_n \to f\}$. So $\mu(E^c) = 0$ and $\int h = \int_E h$ for all $h \in L^1$. Since $|f| \mathbf{1}_E \leq |\lim f_n| \mathbf{1}_E \leq g$, we have that $f \in L^1$.

Suppose first that $(f_n)_n, f$ are real valued. Then, $g - f_n \geq 0$ and $g + f_n \geq 0$ so by Fatou's Lemma,

$$\int g + \int f = \int \liminf (g + f_n) \leq \liminf \int (g + f_n) = \int g + \liminf \int f_n,$$

$$\int g - \int f = \int \liminf (g - f_n) \leq \liminf \int (g - f_n) = \int g - \limsup \int f_n.$$

Subtracting the finite quantity $\int g$ from both sides of both equations we get that

$$\limsup \int f_n \leq \int f \leq \liminf \int f_n,$$

which implies these are equal. □

• **Proposition 10.6.** *If $(f_n)_n$ is a sequence in $L^1(X, \mathcal{F}, \mu)$ such that $\sum_n \int |f_n| d\mu < \infty$, then $f = \sum_n f_n$ is a.e. well defined (the sum converges a.e.), and $\int f d\mu = \sum_n \int f_n d\mu$.*

Proof. Poor man's Fubini tells us that for $g = \sum_n |f_n|$ we have $\int g = \sum_n \int |f_n| < \infty$, so $g \in L^1$. So $N = \{g = \infty\}$ is a null set, and off this set the sum $f = \sum_n f_n$ converges absolutely.

Also, $|\sum_{j=1}^n f_j| \leq g$ for every n. This sequence converges a.e. to f so by Dominated Convergence,

$$\sum_n \int f_n d\mu = \lim_{n\to\infty} \sum_{j=1}^n \int f_j d\mu = \lim_{n\to\infty} \int \sum_{j=1}^n f_j d\mu = \int f d\mu.$$

□

► **Exercise 10.6.** [Folland p.59] Let $(f_n)_n, (g_n)_n, f, g$ all be functions in $L^1(X, \mathcal{F}, \mu)$ such that $f_n \to f$ a.e., $g_n \to g$ a.e., $|f_n| \leq g_n$ a.e., and $\int g_n d\mu \to \int g d\mu$.

Show that $\int f_n d\mu \to \int f d\mu$.

► **Exercise 10.7.** Let $(X, \mathcal{F}, \mu)$ be a measure space and let $(X, \bar{\mathcal{F}}, \bar{\mu})$ be its completion. Show that if $f : X \to \mathbb{C}$ is $\bar{\mathcal{F}}$-measurable, then there exists $g : X \to \mathbb{C}$ such that g is $\mathcal{F}$-measurable and there exists a set $N \in \mathcal{F}$ such that $\mu(N) = 0$ and $\{x : g(x) \neq f(x)\} \subset N$, so $g\mathbf{1}_{N^c} = f\mathbf{1}_{N^c}$ (and so also $g = f$ $\bar{\mu}$-a.e.)

♣ **Solution to ex:10.7.** :(

Recall that

$$\bar{\mathcal{F}} = \{A \cup F \ : \ A \in \mathcal{F}, F \subset N \text{ for some } N \in \mathcal{F} \text{ with } \mu(N) = 0\}.$$

Suppose that $f = \mathbf{1}_{A \cup F}$ for $A \cup F \in \bar{F}$, $A \in \mathcal{F}$, $F \subset N \in \mathcal{F}$ and $\mu(N) = 0$. Then if we set $g = \mathbf{1}_{A \cup N}$ then g is $\mathcal{F}$ measurable, and if $g(x) \neq f(x)$ then $x \in (A \cup N) \setminus (A \cup F) = N \setminus F \subset N$.

Now suppose $f = \sum_k a_k \mathbf{1}_{A_k \cup F_k}$ is a simple function for $X = \biguplus_k (A_k \cup F_k)$, $A_k \in \mathcal{F}$, $F_k \subset N_k \in \mathcal{F}$ with $\mu(N_k) = 0$. Then, defining $g = \sum_k a_k \mathbf{1}_{A_k \cup N_k}$ we have that if $g(x) \neq f(x)$ then $x \in \bigcup_k N_k$ which has μ-measure 0.

If $f \geq 0$ and $\bar{\mathcal{F}}$-measurable, then let $f_n \nearrow f$ be a sequence of $\bar{\mathcal{F}}$-measurable simple functions converging monotony to f. Then for every n there is a set $N_n \in \mathcal{F}$ and a simple $\mathcal{F}$-measurable function g_n such that $\mu(N_n) = 0$ and $g_n \mathbf{1}_{N_n^c} = f_n \mathbf{1}_{N_n^c}$. Taking $N = \bigcup_n N_n$ we have that $\mu(N) = 0$ and $g_n \mathbf{1}_{N^c} = f_n \mathbf{1}_{N^c} \nearrow f \mathbf{1}_{N^c}$. Thus, $g := f \mathbf{1}_{N^c}$ is $\mathcal{F}$-measurable.

Finally if f is any $\bar{\mathcal{F}}$-measurable function, write $f = f_1 - f_2 + i(f_3 - f_4)$ for non-negative $\bar{\mathcal{F}}$-measurable functions $f_1, \ldots, f_4$. For any j, let g_j be a non-negative $\mathcal{F}$-measurable function and $N_j \in \mathcal{F}$ be such that $g_j \mathbf{1}_{N_j^c} = f_j \mathbf{1}_{N_j^c}$ and $\mu(N_j) = 0$. Then for $g := g_1 - g_2 + i(g_3 - g_4)$ and $N := \bigcup_j N_j$ we have that g is $\mathcal{F}$-measurable and $g \mathbf{1}_{N^c} = f \mathbf{1}_{N^c}$ with $\mu(N) = 0$. :) ✓

► **Exercise 10.8.** [Folland, p.59, ex.24] Let $(X, \mathcal{F}, \mu)$ be a measure space of finite measure. Let $(X, \bar{\mathcal{F}}, \bar{\mu})$ be its completion. Let $f : X \to \mathbb{R}$ be a bounded non-negative function.

Show that f is $\bar{\mathcal{F}}$-measurable if and only if there exist sequences $(g_n)_n, (h_n)_n$ of $\mathcal{F}$-measurable simple functions such that $g_n \leq f \leq h_n$ and $\int (h_n - g_n) d\mu < \frac{1}{n}$ for all n.

Show that in this case, $\int f d\bar{\mu} = \lim_{n \to \infty} \int g_n d\mu = \lim_{n \to \infty} \int h_n d\mu$.

♣ **Solution to ex:10.8.** :(

If f is $\bar{\mathcal{F}}$-measurable, then we can find a set $N \in \mathcal{F}$ such that $\mu(N) = 0$ and $f\mathbf{1}_{N^c}$ is $\mathcal{F}$-measurable. Let $M > 0$ be such that $0 \le f \le M$ (as f is bounded). Taking

$$g_n := \tfrac{1}{\mu(X)n}\lfloor \mu(X)nf\mathbf{1}_{N^c}\rfloor \qquad \text{and} \qquad h_n := \tfrac{1}{\mu(X)n}\lceil \mu(X)nf\mathbf{1}_{N^c}\rceil + M\mathbf{1}_N,$$

we have that

$$g_n \le f\mathbf{1}_{N^c} \le f = f\mathbf{1}_{N^c} + f\mathbf{1}_N \le h_n.$$

Also, since $g_n(X), h_n(X) \subset \left\{\frac{1}{\mu(X)n}k \ : \ 0 \le k \le \mu(X)nM, k \in \mathbb{N}\right\}$ we have that g_n, h_n are simple functions. Finally, for $x \notin N$, $(h_n - g_n)(x) \le \frac{1}{\mu(X)n}$ so $\int (h_n - g_n)d\mu = \int_{N^c}(h_n - g_n)d\mu \le n^{-1}$, as required.

For the other direction, assume that g_n, h_n are as in the statement of the exercise. Let $A_n = \left\{h_n - g_n > n^{-1/2}\right\}$.

Note that if $a \le f \le b$ then for every n, either $a - n^{-1/2} \le g_n \le h_n \le b + n^{-1/2}$ or $a \le f \le b$ and $h_n - g_n > n^{-1/2}$. Also, if for every n, $a - n^{-1/2} \le g_n \le h_n \le b + n^{-1/2}$, then

$$a = \lim_n (a - n^{-1/2}) \le f \le \lim_n (b + n^{-1/2}) = b.$$

Thus,

$$\{a \le f \le b\} = \bigcap_n \left(\left\{a - n^{-1/2} \le g_n \le h_n \le b + n^{-1/2}\right\} \bigcup (\{a \le f \le b\} \cap A_n)\right).$$

Since $\left\{a - n^{-1/2} \le g_n \le h_n \le b + n^{-1/2}\right\} \in \mathcal{F}$ we only need to show that the set $\bigcap_n A_n$ has μ-measure 0. So it suffices to show that $\inf_n \mu(A_n) = 0$. To this end,

$$\mu(A_n) \le \int_{A_n} \sqrt{n} \cdot (h_n - g_n)d\mu \le \sqrt{n} \cdot n^{-1} \to 0.$$

Thus, $\{a \le f \le b\}$ is the union of a $\mathcal{F}$-measurable set and a subset of a μ-null set, and so f is $\bar{\mathcal{F}}$-measurable.

Now if $g_n \le f \le h_n$ as in the statement of the exercise, since f is $\bar{\mathcal{F}}$-measurable, we may find a set $N \in \mathcal{F}$ such that $\mu(N) = 0$ and $f\mathbf{1}_{N^c}$ is $\mathcal{F}$-measurable. By replacing g_n with $\sup_{k \le n} g_k$ we may assume without loss of generality that $g_n\mathbf{1}_{N^c} \nearrow f\mathbf{1}_{N^c}$, so by monotone convergence, $\int g_n d\mu \to \int f\mathbf{1}_{N^c}d\mu$ and $\int g_n d\bar{\mu} \to \int f d\bar{\mu}$. Also,

$$\int g_n d\mu \le \int h_n d\mu \le \int g_n d\mu + n^{-1}.$$

So $\int h_n d\mu \to \int f \mathbf{1}_{N^c} d\mu$ as well.

Hence we are left with showing that $\int f \mathbf{1}_{N^c} d\mu = \int f d\bar{\mu}$, and for this it suffices to prove that for all n, $\int g_n d\mu = \int g_n d\bar{\mu}$. Because g_n are simple functions and $\mathcal{F}$-measurable for all n, by linearity it suffices that for any set $A \in \mathcal{F}$, we have $\int \mathbf{1}_A d\mu = \mu(A) = \bar{\mu}(A) = \int \mathbf{1}_A d\bar{\mu}$. :) ✓

10.4. Riemann vs. Lebesgue integration

Recall the Riemann integral of a bounded function $f : [a, b] \to \mathbb{R}$:

For every partition $P = (x_k)_{k=0}^n$, $a = x_0 < x_1 < \cdots < x_n = b$, define

$$S_P f = \sum_{k=1}^n \sup_{x \in [x_{k-1}, x_k]} f(x) \cdot (x_k - x_{k-1}) \qquad \text{and} \qquad s_P f = \sum_{k=1}^n \inf_{x \in [x_{k-1}, x_k]} f(x) \cdot (x_k - x_{k-1}).$$

Define $I_a^b f = \inf S_P f$ and $i_a^b f = \sup s_P f$ where the infimum and supremum are over all partitions P of $[a, b]$.

If $I_a^b f = i_a^b f$ we say that f is Riemann integrable and define $\int_a^b f dx$ to be the common value.

Given a partition of $[a, b]$, $P = (x_k)_{k=0}^n$, define the functions

$$\Psi_P f = \sum_{k=1}^n \sup_{x \in [x_{k-1}, x_k]} f(x) \cdot \mathbf{1}_{[x_{k-1}, x_k]} \qquad \text{and} \qquad \psi_P f = \sum_{k=1}^n \inf_{x \in [x_{k-1}, x_k]} f(x) \cdot \mathbf{1}_{[x_{k-1}, x_k]}.$$

Note that

$$S_P f = \int \Psi_P f d\lambda \qquad \text{and} \qquad s_P f = \int \psi_P f d\lambda.$$

If f is Riemann integrable, we may choose a sequence of partitions $(P_k)_k$ such that $\lim S_{P_k} f = \lim s_{P_k} f = \int_a^b f dx$. We may also choose the partitions as refinements of the previous ones in the sequence. In this case the functions $\Psi_k := \Psi_{P_k} f$ form a decreasing sequence and $\psi_k := \psi_{P_k} f$ form an increasing sequence. So there exist limiting functions $\Psi = \lim \Psi_k$ and $\psi = \lim \psi_k$. Note that $\psi \leq f \leq \Psi$.

Since f is bounded, and since $\lambda([a, b]) < \infty$, we get that $(\psi_k)_k, (\Psi_k)_k$ are dominated sequences, and so by Dominated Convergence,

$$\int \psi d\lambda = \int \Psi d\lambda = \int_a^b f dx.$$

So $\int |\Psi - \psi| d\lambda = \int (\Psi - \psi) d\lambda = 0$ and so $\Psi = \psi$ a.e., which implies that $\Psi = f = \psi$ a.e.

Thus, $f\mathbf{1}_A = \Psi\mathbf{1}_A$ for some A such that $\lambda(A^c) = 0$. Since λ is a complete measure on $(\mathbb{R}, \mathcal{L})$, we have that f is Lebesgue measurable. Also,

$$\int_{[a,b]} f d\lambda = \int_{[a,b]} \Psi d\lambda = \int_a^b f dx.$$

Conclusion: If f is Riemann integrable on $[a, b]$ then it is also Lebesgue measurable on $[a, b]$ and $\int_{[a,b]} f d\lambda = \int_a^b f dx$.

► **Exercise 10.9.** Given a bounded function $f : [a, b] \to \mathbb{R}$ define

$$\Theta(x) = \lim_{n\to\infty} \sup_{|y-x|\le n^{-1}} f(y) \qquad \text{and} \qquad \theta(x) = \lim_{n\to\infty} \inf_{|y-x|\le n^{-1}} f(y).$$

Show that f is continuous at x if and only if $\Theta(x) = \theta(x)$.

Show that a.e.

$$\int_{[a,b]} \Theta d\lambda = I_a^b f \qquad \text{and} \qquad \int_{[a,b]} \theta d\lambda = i_a^b f.$$

Conclude that f is Reimann integrable if and only if

$$\lambda(\{x \ : \ f \text{ is not continuous at } x\}) = 0.$$

♣ **Solution to ex:10.9.** :(

Write

$$\Theta_n(x) = \sup_{|y-x|\le n^{-1}} f(y) \qquad \text{and} \qquad \theta_n(x) = \inf_{|y-x|\le n^{-1}} f(y).$$

Assume that $\Theta(x) = \theta(x)$. Let $\varepsilon > 0$. Choose n large enough so that $|\Theta_n(x)-\Theta(x)| < \varepsilon/2$ and $|\theta_n(x) - \theta(x)| < \varepsilon/2$. So if $|y - x| < \frac{1}{n}$ we have that

$$\theta(x) - \varepsilon/2 < \theta_n(x) \le f(y) \le \Theta_n(x) < \Theta(x) + \varepsilon/2.$$

Since $\Theta(x) - \theta(x)$, for all $|y - x| < \frac{1}{n}$ we have that $|f(x) - f(y)| < \varepsilon$.

That is, for all $\varepsilon > 0$ there exists n large enough so that for all $|y - x| < \frac{1}{n}$ we have $|f(y) - f(x)| < \varepsilon$. So f is continuous at x.

For the other direction, if f is continuous at x, then for any $\varepsilon > 0$ there exists large enough n such that we have for all $|y - x| < \frac{1}{n}$ that $|f(y) - f(x)| < \varepsilon$. Thus, for all $\varepsilon > 0$ there exists n_0 such that if $n > n_0$ we have $|\Theta_n(x) - \theta_n(x)| < \varepsilon$. Since $\Theta_n(x) \to \Theta(x), \theta_n(x) \to \theta(x)$, we get that $\Theta(x) = \theta(x)$.

Now, consider a partition $P = (x_k)_{k=0}^m$. Let $|P| = \max_k |x_k - x_{k-1}|$. If $2|P| < \frac{1}{n}$, for any $x \in [x_{k-1}, x_k]$, we have that

$$\sup_{y \in [x_{k-1}, x_k]} f(y) \leq \Theta_n(x) \qquad \text{and} \qquad \inf_{y \in [x_{k-1}, x_k]} f(y) \geq \theta_n(x).$$

Thus,

$$I_a^b f \leq S_P f = \sum_{k=1}^m \sup_{y \in [x_{k-1}, x_k]} f(y) \int_{[x_{k-1}, x_k]} d\lambda \leq \int_{[a,b]} \Theta_n d\lambda,$$

and similarly,

$$i_a^b f \geq s_P f = \sum_{k=1}^m \inf_{y \in [x_{k-1}, x_k]} f(y) \int_{[x_{k-1}, x_k]} d\lambda \geq \int_{[a,b]} \theta_n d\lambda.$$

On the other hand, if n is very large and $x \in [x_{k-1} + \frac{1}{n}, x_k - \frac{1}{n}]$ then

$$\Theta_n(x) \leq \sup_{y \in [x_{k-1}, x_k]} f(y) \qquad \text{and} \qquad \theta_n(x) \geq \inf_{y \in [x_{k-1}, x_k]} f(y).$$

So since f is bounded, say by $|f| \leq M$, we have that

$$\int_{[x_{k-1}, x_k]} \Theta_n d\lambda \leq \sup_{y \in [x_{k-1}, x_k]} f(y) \cdot (x_k - x_{k-1}) + M \cdot \tfrac{2}{n},$$

and

$$\int_{[a,b]} \Theta_n d\lambda \leq S_P f + m \cdot M \cdot \tfrac{2}{n},$$

where m is the number of intervals in the partition. Similarly,

$$\int_{[a,b]} \theta_n d\lambda \geq s_P f - m \cdot M \cdot \tfrac{2}{n}.$$

By Dominated Convergence, taking $n \to \infty$ we have that for any partition P,

$$\int_{[a,b]} \Theta d\lambda \leq S_P f \qquad \text{and} \qquad \int [a,b] \theta d\lambda \geq s_P f.$$

Taking infimum and supremum respectively, we have that

$$\int_{[a,b]} \Theta d\lambda = I_a^b f \qquad \text{and} \qquad \int_{[a,b]} \theta d\lambda = i_a^b f.$$

Finally, note that f is Riemann integrable if and only if $\int_{[a,b]}(\Theta - \theta)d\lambda = 0$ which is if and only if $\Theta = \theta$ a.e. on $[a,b]$, which is if and only if f is continuous λ-a.e. on $[a,b]$. :) ✓

▶ **Exercise 10.10.** Let $f \in L^1(\mathbb{R}, \mathcal{B}, \mu)$ and $F(x) := \int_{(-\infty,x]} f d\mu$.

- Show that if $f \geq 0$ then F is right continuous and F is continuous at x if and only if $f(x)\mu(\{x\}) = 0$.
- Show that if $\mu(\{x\}) = 0$ then F is continuous at x.

♣ **Solution to ex:10.10.** :(

For any x and ε, note that

$$F(x+\varepsilon) - F(x) = \int_{(x,x+\varepsilon]} f d\mu.$$

First assume that f is non-negative. Then, $\nu(A) := \int_A f d\mu$ is a measure. If $x_n \searrow x$ then the sets $A_n = (x, x_n]$ are decreasing and

$$F(x_n) - F(x) = \nu(A_n) \to \nu(\bigcap_n A_n) = \nu(\emptyset) = 0.$$

So F is right continuous.

If $x_n \nearrow x$ then

$$F(x) - F(x_n) \to \nu(\{x\}) = \int_{\{x\}} f d\mu = f(x)\mu(\{x\}).$$

because $\bigcap_n (x_n, x] = \{x\}$. So F is left continuous as well if and only if $f(x)\mu(\{x\}) = 0$.

This resolves the case where f is non-negative.

For general $f \in L^1(\mu)$, write $f = f_1 - f_2 + i(f_3 - f_4)$ for non-negative f_j. Then $F = F_1 - F_2 + i(F_3 - F_4)$ where $F_j(x) = \int_{(-\infty,x]} f_j d\mu$.

If $\mu(\{x\}) = 0$ then for every j the function F_j is continuous at x, and thus so is F. :) ✓

Number of exercises in lecture: 10

Total number of exercises until here: 106

Measure Theory

Ariel Yadin

Lecture 11: Product measures

Recall that if $(X,\mathcal{F}),(Y,\mathcal{G})$ are measurable spaces then $\mathcal{F}\otimes\mathcal{G}=\sigma(A\times B\ :\ A\in\mathcal{F},B\in\mathcal{G})$ is a σ-algebra on $X\times Y$.

An example is $\mathcal{B}_1\otimes\mathcal{B}_1=\mathcal{B}_2$.

When we constructed Lebesgue measure in $\mathbb{R}^d$ "hands on" we took boxes as basic building blocks; these are just products of intervals which where the basic building blocks in dimension 1. We now give a general construction of a product measure on $(X,\mathcal{F})\otimes(Y,\mathcal{G})$.

• **Definition 11.1.** For measurable spaces $(X,\mathcal{F}),(Y,\mathcal{G})$, a **box** is a set of the form $A\times B$ where $A\in\mathcal{F},B\in\mathcal{G}$.

► **Exercise 11.1.** Show that for $A,C\subset X$ and $B,D\subset Y$ we have

$$(A\times B)\cap(C\times D)=(A\cap C)\times(B\cap D)\qquad\text{and}\qquad (A\times B)^c=(A^c\times Y)\uplus(A\times B^c)=(X\times B^c)\uplus(A^c\times B).$$

► **Exercise 11.2.** Show that the family of all finite disjoint unions of boxes (from $(X,\mathcal{F}),(Y,\mathcal{G})$ forms an algebra.

Show that the σ-algebra generated by this algebra is $\mathcal{F}\otimes\mathcal{G}$.

► **Exercise 11.3.** Show that $\mathbf{1}_{A\times B}(x,y)=\mathbf{1}_A(x)\mathbf{1}_B(y)$.

• **Proposition 11.2.** *Let $(X, \mathcal{F}, \mu), (Y, \mathcal{G}, \nu)$ be measure spaces.*

If $A \times B = \biguplus_n (A_n \times B_n)$ where $A, (A_n)_n$ are $\mathcal{F}$-measurable and $B, (B_n)_n$ are $\mathcal{G}$-measurable, then

$$\mu(A)\nu(B) = \sum_n \mu(A_n)\nu(B_n).$$

Proof. This is actually just poor man's Fubini.

We have by the assumption $A \times B = \biguplus_n (A_n \times B_n)$ that

$$\mathbf{1}_A(x)\mathbf{1}_B(y) = \sum_n \mathbf{1}_{A_n}(x)\mathbf{1}_{B_n}(y).$$

Integrate this function over X for fixed y to get that for all y,

$$\mu(A)\mathbf{1}_B(y) = \sum_n \mu(A_n)\mathbf{1}_{B_n}(y),$$

where we have used poor man's Fubini once. Integrate this again over Y to get by poor man's Fubini again,

$$\mu(A)\nu(B) = \sum_n \mu(A_n)\nu(B_n).$$

□

▶ **Exercise 11.4.** Let $(X, \mathcal{F}, \mu), (Y, \mathcal{G}, \nu)$ be measure spaces.

Show that if

$$\biguplus_n (A_n \times B_n) = \biguplus_n (C_n \times D_n)$$

where $A_n, C_n \in \mathcal{F}, B_n, D_n \in \mathcal{G}$, then

$$\sum_n \mu(A_n)\nu(B_n) = \sum_n \mu(C_n)\nu(D_n).$$

♣ **Solution to ex:11.4.** :(

We have

$$(A_j \times B_j) \cap (C_k \times D_k) = (A_j \cap C_k) \times (B_j \times D_k),$$

so

$$A_j \times B_j = \biguplus_n (A_j \cap C_n) \times (B_j \times D_n),$$

which implies that

$$\mu(A_j)\nu(B_j) = \sum_n \mu(A_j \cap C_n)\nu(B_j \cap D_n).$$

Similarly,

$$\mu(C_k)\nu(D_k) = \sum_n \mu(A_n \cap C_k)\nu(B_n \cap D_k).$$

Thus,

$$\sum_n \mu(A_n)\nu(B_n) = \sum_{n,k} \mu(A_n \cap C_k)\nu(B_n \cap D_k) = \sum_k \mu(C_k)\nu(D_k).$$

:) ✓

► **Exercise 11.5.** Let $E = \biguplus_{j=1}^n (A_j \times B_j) \in \mathcal{A}$. Define $\rho(E) = \sum_{j=1}^n \mu(A_j)\nu(B_j)$. Then ρ is well defined, and defines a pre-measure on $\mathcal{A}$.

By Charathéodory's Extension Theorem we may extend ρ above to a measure on all of $\sigma(\mathcal{A}) = \mathcal{F} \otimes \mathcal{G}$. Also note that if X and Y are σ-finite with respect to μ and ν, then $X \times Y$ is σ-finite with respect to ρ. So in this case the extension, denoted $\mu \otimes \nu$, is the unique measure on $(X, \mathcal{F}) \otimes (Y, \mathcal{G})$ satisfying

$$\mu \otimes \nu(A \times B) = \mu(A)\nu(B) \qquad \forall\ A \in \mathcal{F}, B \in \mathcal{G}.$$

► **Exercise 11.6.** Show that if X and Y are σ-finite with respect to μ and ν, then $X \times Y$ is σ-finite with respect to ρ.

♣ **Solution to ex:11.6.** :(

If $X = \biguplus_n A_n, Y = \biguplus_n B_n$ where $\mu(A_n) < \infty, \nu(B_n) < \infty$, then $X \times Y = \biguplus_{n,k}(A_n \times B_k)$ and $\rho(A_n \times B_k) = \mu(A_n)\nu(B_k) < \infty$. :) ✓

▶ **Exercise 11.7.** In this exercise the product measure is generalized to further products. Let $(X_j, \mathcal{F}_j, \mu_j)$ be σ-finite measure spaces, $j = 1, 2, \ldots$.

Show that $(\mu_1 \otimes \mu_2) \otimes \mu_3 = \mu_1 \otimes (\mu_2 \otimes \mu_3)$.

Show that for all n there exists a unique measure $\bigotimes_{j=1}^n \mu_j$ on $\bigotimes_{j=1}^n (X_j, \mathcal{F}_j)$ satisfying that for all $A_j \in \mathcal{F}_j$, $j = 1, 2, \ldots, n$,

$$\bigotimes_{j=1}^n \mu_j(A_1 \times \cdots \times A_n) = \prod_{j=1}^n \mu_j(A_j).$$

11.1. Sections

• **Definition 11.3.** Let $E \subset X \times Y$. Define the **sections** of E for every $x \in X, y \in Y$ as

$$E_x = \{y \in Y \ : \ (x,y) \in E\} = \pi_Y(E \cap (\{x\} \times Y))$$
$$E^y = \{x \in X \ : \ (x,y) \in E\} = \pi_X(E \cap (X \times \{y\})),$$

where $\pi_X : X \times Y \to X, \pi_Y : X \times Y \to Y$ are the natural projections.

For a function $f : X \times Y \to Z$ define the sections of f for every $x \in X, y \in Y$ as the functions $f_x : Y \to Z, f^y : X \to Z$ by the formula $f_x(y) = f^y(x) = f(x,y)$.

▶ **Exercise 11.8.** Show that $(\mathbf{1}_E)_x = \mathbf{1}_{E_x}$ and $(\mathbf{1}_E)^y = \mathbf{1}_{E^y}$.

Show that $(f_x)^{-1}(A) = (f^{-1}(A))_x$ and $(f^y)^{-1}(A) = (f^{-1}(A))^y$.

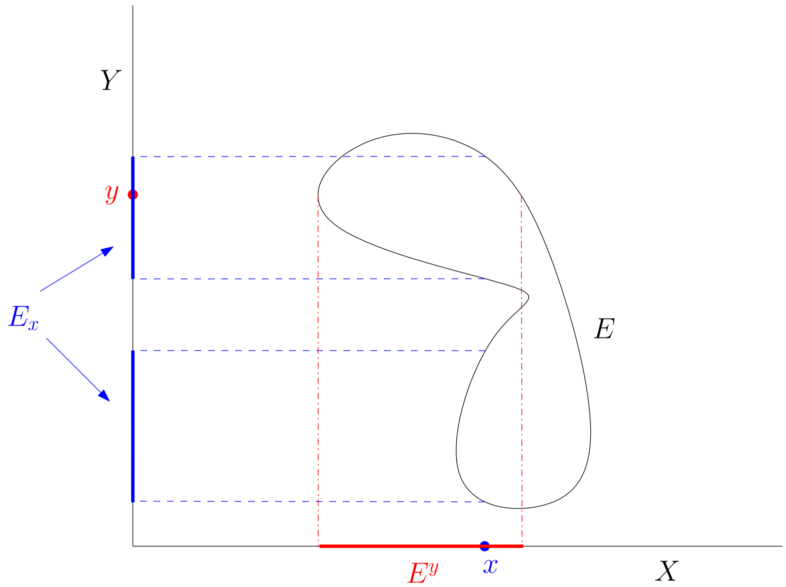

FIGURE 4. Depiction of E_x, E^y.

♣ **Solution to ex:11.8.** :(

We have for all x, y,

$$(\mathbf{1}_E)_x(y) = \mathbf{1}_{\{(x,y)\in E\}} = \mathbf{1}_{\{y \in E_x\}} = \mathbf{1}_{E_x}(y),$$

and similarly for $(\mathbf{1}_E)^y = \mathbf{1}_{E^y}$.

Also,

$$y \in (f_x)^{-1}(A) \iff f_x(y) \in A \iff f(x,y) \in A \iff (x,y) \in f^{-1}(A) \iff y \in (f^{-1}(A))_x,$$

and similarly for $(f^y)^{-1}(A) = (f^{-1}(A))^y$. :) ✓

► **Exercise 11.9.** Show that $(\bigcup_n E_n)_x = \bigcup_n (E_n)_x$ and $(\bigcup_n E_n)^y = \bigcup_n (E^y_)$.
Show that $(E^c)_x = (E_x)^c$ and $(E^c)^y = (E^y)^c$.

• **Proposition 11.4.** *Let $(X, \mathcal{F}), (Y, \mathcal{G})$ be measurable spaces. If $E \in \mathcal{F} \otimes \mathcal{G}$ then for all $x \in X, y \in Y$, $E^y \in \mathcal{F}$ and $E_x \in \mathcal{G}$.*

Also if f is a function on $X \times Y$ that is $\mathcal{F} \otimes \mathcal{G}$-measurable, then f^y is $\mathcal{F}$-measurable and f_x is $\mathcal{G}$-measurable.

Proof. Let

$$\mathcal{H} = \{E \subset X \times Y \ : \ \forall\, x \in X, y \in Y \ , \ E^y \in \mathcal{F}, E_x \in \mathcal{G}\} .$$

One may verify that $\mathcal{H}$ is a σ-algebra (exercise!).

If $A \times B \in \mathcal{F} \otimes \mathcal{G}$ is a box then $(A \times B)_x = B$ for $x \in A$, and $(A \times B)^y = A$ for $y \in B$, and $(A \times B)_x = \emptyset = (A \times B)^y$ if $x \notin A$ or $y \notin B$. So all boxes are in $\mathcal{H}$. Thus, $\mathcal{F} \otimes \mathcal{G} \subset \mathcal{H}$, which proves the first assertion.

If f is $\mathcal{F} \otimes \mathcal{G}$-measurable then for any set A in the target σ-algebra, $(f_x)^{-1}(A) = (f^{-1}(A))_x$ which is in $\mathcal{G}$ because $f^{-1}(A) \in \mathcal{F} \otimes \mathcal{G}$. Similarly for $(f^y)^{-1}(A) = (f^{-1}(A))^y \in \mathcal{F}$. ☐

11.2. Product integrals

11.2.1. Monotone classes and algebras.

We require some technical notions.

✓ A **monotone class** on X is a family $\mathcal{C} \subset 2^X$ that is closed under countable increasing unions and countable decreasing intersections; *i.e.* for $(E_n)_n \subset \mathcal{C}$, if $E_n \subset E_{n+1}$ for all n then $\bigcup_n \in \mathcal{C}$, and if $E_n \supset E_{n+1}$ for all n then $\bigcap_n \in \mathcal{C}$.

► **Exercise 11.10.** Show that an intersection of monotone classes is a monotone class.

Show that for any collection of set $\mathcal{K} \subset 2^X$ there is a monotone class $\mathcal{C}(\mathcal{K})$ that is the minimal monotone class containing $\mathcal{K}$; *i.e.* $\mathcal{K} \subset \mathcal{C}(\mathcal{K})$ and for any monotone class $\mathcal{C}$ such that $\mathcal{K} \subset \mathcal{C}$ we have $\mathcal{C}(\mathcal{K}) \subset \mathcal{C}$.

► **Exercise 11.11.** Show that any σ-algebra is a monotone class.

Show that if $\mathcal{C}$ is a monotone class and an algebra, then $\mathcal{C}$ is a σ-algebra.

♣ **Solution to ex:11.11.** :(

A σ-algebra is closed under countable unions and intersections, so this is much more than is needed for it to be a monotone class. This proves the first assertion.

Now suppose that $\mathcal{C}$ is a monotone class and an algebra. Let $(A_n)_n$ be a sequence in $\mathcal{C}$. We need to prove that $\bigcup_n A_n \in \mathcal{C}$.

Indeed, set $B_n = \bigcup_{j=1}^n A_j$. Because $\mathcal{C}$ is an algebra, $(B_n)_n$ is an increasing sequence in $\mathcal{C}$. Because $\mathcal{C}$ is a monotone class, $\bigcup_n A_n = \bigcup_n B_n \in \mathcal{C}$. :) ✓

► **Exercise 11.12.** Let $\mathcal{C}$ be a monotone class. Fix $A \in \mathcal{C}$ and define

$$\mathcal{C}_A = \{B \in \mathcal{C} \ : \ A \setminus B, B \setminus A, A \cap B \text{ are all in } \mathcal{C}\}.$$

Show that $\mathcal{C}_A \subset \mathcal{C}$ is a monotone class.

♣ **Solution to ex:11.12.** :(

If $(B_n)_n$ is an increasing sequence in $\mathcal{C}_A$ then $(B_n \setminus A)_n, (B_n \cap A)_n$ are increasing sequences in $\mathcal{C}$. So,

$$\bigcup_n B_n \setminus A = \bigcup_n (B_n \setminus A) \in \mathcal{C} \qquad \text{and} \qquad \bigcup_n B_n \cap A = \bigcup_n (B_n \cap A) \in \mathcal{C}.$$

Also, $(A \setminus B_n)_n$ is a decreasing sequence in $\mathcal{C}$ so

$$A \setminus \bigcup_n B_n = A \cap \bigcap_n B_n^c = \bigcap_n (A \setminus B_n) \in \mathcal{C}.$$

Thus, $\bigcup_n B_n \in \mathcal{C}_A$ as well. Similarly, if $(B_n)_n$ is a decreasing sequence in $\mathcal{C}_A$, then $(B_n \setminus A)_n, (B_n \cap A)_n$ are decreasing sequences in $\mathcal{C}$ and $(A \setminus B_n)_n$ is an increasing sequence in $\mathcal{C}$, which leads to

$$\bigcap_n B_n \setminus A = \bigcap_n (B_n \setminus A) \in \mathcal{C} \qquad \text{and} \qquad \bigcap_n B_n \cap A = \bigcap_n (B_n \cap A) \in \mathcal{C},$$

and

$$A \setminus \bigcap_n B_n = A \cap \bigcup_n B_n^c = \bigcup_n (A \setminus B_n) \in \mathcal{C}.$$

So $\bigcap_n B_n \in \mathcal{C}_A$ in this case. :) ✓

• **Lemma 11.5** (Monotone Class Lemma)**.** *For an algebra $\mathcal{A}$ the monotone class $\mathcal{C}(\mathcal{A}) = \sigma(\mathcal{A})$.*

Proof. Since $\sigma(\mathcal{A})$ is a monotone class continuing $\mathcal{A}$, we have that $\mathcal{C}(\mathcal{A}) \subset \sigma(\mathcal{A})$. So it suffices to prove the other inclusion. Since $\mathcal{A} \subset \mathcal{C}(\mathcal{A})$ it suffices to show that $\mathcal{C}(\mathcal{A})$ is a σ-algebra.

For $A \in \mathcal{C}(\mathcal{A})$ define

$$\mathcal{C}_A = \{B \in \mathcal{C}(\mathcal{A}) \ : \ A \setminus B, B \setminus A, A \cap B \text{ are all in } \mathcal{C}(\mathcal{A})\}.$$

We have that $\mathcal{C}_A$ is a monotone class.

If $A \in \mathcal{A}$ then since $\mathcal{A}$ is an algebra we have that $\mathcal{A} \subset \mathcal{C}_A$. By the minimality of $\mathcal{C}(\mathcal{A})$ we get that $\mathcal{C}_A = \mathcal{C}(\mathcal{A})$ for all $A \in \mathcal{A}$.

Note that by definition, $B \in \mathcal{C}_A$ if and only if $A \in \mathcal{C}_B$. This implies that for all $A \in \mathcal{A}$ and any $B \in \mathcal{C}(\mathcal{A}) = \mathcal{C}_A$ we have that $A \in \mathcal{C}_B$. Thus, $\mathcal{A} \subset \mathcal{C}_B$ for any $B \in \mathcal{C}(\mathcal{A})$. By minimality of $\mathcal{C}(\mathcal{A})$ again, $\mathcal{C}_B = \mathcal{C}(\mathcal{A})$ for any $B \in \mathcal{C}(\mathcal{A})$.

We conclude that if $A, B \in \mathcal{C}(\mathcal{A})$ then $A \setminus B, B \setminus A, A \cap B$ are all in $\mathcal{C}(\mathcal{A})$. Since $X = \emptyset^c \in \mathcal{A} \subset \mathcal{C}(\mathcal{A})$ we have that $\mathcal{C}(\mathcal{A})$ is closed under complements as well. So $\mathcal{C}(\mathcal{A})$ is an algebra and a monotone class. Thus $\mathcal{C}(\mathcal{A})$ is a σ-algebra, and we are done. ☐

► **Exercise 11.13.** Let $(X, \mathcal{F})$ be a measurable space such that $\mathcal{F} = \sigma(\mathcal{A})$ where $\mathcal{A}$ is an algebra. Let L^∞ be the vector space of all bounded measurable functions from X to $\mathbb{R}$. Let V be a subspace of L^∞ such that:

- $1 \in V$ and $\mathbf{1}_A \in V$ for all $A \in \mathcal{A}$.
- If $(f_n)_n$ is a monotone increasing sequence in V such that $f_n \nearrow f$ and such that f is bounded, then $f \in V$ as well.

Show that $V = L^\infty$.

♣ **Solution to ex:11.13.** :(

Define $\mathcal{G} = \{A \ : \ \mathbf{1}_A \in V\}$. By the assumptions, $\mathcal{A} \subset \mathcal{G}$. Also, if $(A_n)_n$ is an increasing sequence in $\mathcal{G}$, then $(\mathbf{1}_{A_n})_n$ is an increasing sequence in V such that $\mathbf{1}_{A_n} \nearrow \mathbf{1}_A$ where $A = \bigcup_n A_n$. Since $\mathbf{1}_A$ is bounded, we have that $\mathbf{1}_A \in V$, which implies that $A \in \mathcal{G}$. Note that $\mathcal{G}$ is closed under complements: If $A \in \mathcal{G}$ then $\mathbf{1}_A \in V$ and so $\mathbf{1}_{A^c} = 1 - \mathbf{1}_A \in V$ (because V is a vector space). Thus, if $(A_n)_n$ is a decreasing sequence in $\mathcal{G}$, then $(A_n^c)_n$ is an increasing sequence in $\mathcal{G}$, and $\bigcap_n A_n = \left(\bigcup_n A_n^c\right)^c \in \mathcal{G}$.

We conclude that $\mathcal{G}$ is a monotone class. Thus, by the Monotone Class Lemma, $\mathcal{F} = \mathcal{C}(\mathcal{A}) \subset \mathcal{G}$.

So $\mathbf{1}_A \in V$ for all $A \in \mathcal{F}$. Since V is a vector space, all simple functions are also in V.

Now, if $f \geq 0$ is a bounded measurable function, then there exists a monotone sequence of simple functions $\varphi_n \nearrow f$. Since f is bounded, the assumptions on V tell us that $f \in V$. So V contains all non-negative bounded measurable functions.

For a general real-valued bounded measurable function, note that $f^+, f^- \in V$ so also $f = f^+ - f^-$. :) ✓

11.2.2. **Fubini-Tonelli for indicators.** The next theorem tells us how to compute the product measure of a set. First measure each section in one axis, and then integrate all measures of sections over the other axis.

••• **Theorem 11.6.** *Let $(X, \mathcal{F}, \mu), (Y, \mathcal{G}, \nu)$ be σ-finite measure spaces. For any $E \in \mathcal{F} \otimes \mathcal{G}$ we have that the functions*

$$x \mapsto \nu(E_x) \qquad \textit{and} \qquad y \mapsto \mu(E^y)$$

are measurable. Moreover,

$$\mu \otimes \nu(E) = \int \nu(E_x) d\mu = \int \mu(E^y) d\nu.$$

Proof. **Step I.** First assume that μ, ν are *finite* measures.

Define $\mathcal{C}$ to be the set of all set $E \in \mathcal{F} \otimes \mathcal{G}$ such that the theorem holds for E. We need to show that $\mathcal{C} = \mathcal{F} \otimes \mathcal{G}$.

If $E = A \times B$ for $A \in \mathcal{F}, B \in \mathcal{G}$ then $E_x = B$ and $E^y = A$ for $(x,y) \in E$ and $E_x = \emptyset = E^y$ for $(x,y) \notin E$. So $x \mapsto \nu(E_x)$ is the function $\nu(B)\mathbf{1}_A$ and $y \mapsto \mu(E^y)$ is the function $\mu(A)\mathbf{1}_B$. These are of course measurable. Also, $\mu \otimes \nu(A \times B) = \mu(A)\nu(B)$ by definition, which is

$$\mu \otimes \nu(A \times B) = \int \nu(B)\mathbf{1}_A d\mu = \int \mu(A)\mathbf{1}_B d\nu.$$

Thus, all boxes $A \times B$ are in $\mathcal{C}$.

We have already seen that if $E = \biguplus_{j=1}^n (A_j \times B_j)$ then

$$\mu \otimes \nu(E) = \sum_{j=1}^n \mu(A_j)\nu(B_j).$$

Since in this case

$$E_x = \biguplus_{j=1}^n (A_j \times B_j)_x \qquad \text{and} \qquad E^y = \biguplus_{j=1}^n (A_j \times B_j)^y,$$

the functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E_y)$ are just $\sum_{j=1}^n \nu(B_j)\mathbf{1}_{A_j}$ and $\sum_{j=1}^n \mu(A_j)\mathbf{1}_{B_j}$, which are measurable and satisfy the conclusion of the theorem by additivity of the integral.

Thus all finite disjoint unions of boxes are in $\mathcal{C}$, which is to say that $\mathcal{C}$ contains the algebra that generates $\mathcal{F} \otimes \mathcal{G}$. If $\mathcal{C}$ was a monotone class then we would have by the Monotone Class Lemma that $\mathcal{F} \otimes \mathcal{G} \subset \mathcal{C}$.

So we are left with showing that $\mathcal{C}$ is a monotone class.

To this end, let $(E_n)_n$ be an increasing sequence in $\mathcal{C}$. Let $E = \bigcup_n E_n$. Define $f_n(x) = \nu((E_n)_x)$. This is a measurable function satisfying $\int f_n d\mu = \mu \otimes \nu(E_n)$. Also, $(f_n)_n$ form an increasing sequence in $L^+(X, \mathcal{F}, \mu)$, because $((E_n)_x)_n$ is an increasing sequence. Note that $E_x = \bigcup_n (E_n)_x$ which implies that $f_n \nearrow f$ for $f(x) = \nu(E_x)$ by continuity of measures. So $f \in L^+(X, \mathcal{F}, \mu)$. By Monotone Convergence and continuity of the measure $\mu \otimes \nu$,

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu = \lim_{n\to\infty} \mu \otimes \nu(E_n) = \mu \otimes \nu(E).$$

Similarly, the functions $g_n(y) = \mu((E_n)^y)$ converge monotonically $g_n \nearrow g$ for $g(y) = \mu(E^y)$, and using Monotone Convergence as before,

$$\int g d\nu = \mu \otimes \nu(E).$$

Thus, $E \in \mathcal{C}$.

Now for the case of a decreasing sequence $(E_n)_n$ in $\mathcal{C}$. Let $E = \bigcap_n E_n$. Again we define $f_n(x) = \nu((E_n)_x)$ and $g_n(y) = \mu((E_n)^y)$. These are decreasing sequences that converge to $f(x) = \nu(E_x)$ and $g(y) = \mu(E^y)$ respectively. So f, g are measurable. Also, for all n, $|f_n| \le \nu(Y) < \infty$ and $|g_n| \le \mu(X) < \infty$. Since the measures are finite, constant functions are in L^1 for both μ and ν. Thus, we may use Dominated Convergence of conclude that

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu = \lim_{n\to\infty} \mu \otimes \nu(E_n) = \mu \otimes \nu(E),$$

and similarly,

$$\int g d\nu = \mu \otimes \nu(E).$$

So $E \in \mathcal{C}$ in the decreasing case as well.

This shows that $\mathcal{C}$ is a monotone class, completing the proof in the case μ, ν are finite measures.

Step II. For general μ, ν which are σ-finite, let $X \times Y = \bigcup_n (X_n \times Y_n)$ where $(X_n)_n, (Y_n)_n$ are increasing sequences of measurable sets, and $\mu(X_n) < \infty, \nu(Y_n) < \infty$.

Let $E \in \mathcal{F} \otimes \mathcal{G}$. For every n, consider $E \cap (X_n \times Y_n)$. The functions $\mu_n(A) = \mu(A \cap X_n), \nu_n(B) = \nu(B \cap Y_n)$ are finite measures. For any box $A \times B$ we have that $\mu_n \otimes \nu_n(A \times B) = \mu_n(A)\nu_n(B) = \mu \otimes \nu(A \times B \cap X_n \times Y_n)$. Thus, $\rho_n(C) := \mu \otimes \nu(C \cap X_n \times Y_n)$ is the unique σ-finite product measure $\rho_n = \mu_n \otimes \nu_n$ on $\mathcal{F} \otimes \mathcal{G}$. By the previous step, the functions $x \mapsto \nu_n(E_x) = \nu(E_x \cap Y_n)$ and $y \mapsto \mu_n(E^y) = \mu(E^y \cap X_n)$ are measurable. Since $(X_n)_n, (Y_n)_n$ are increasing sequences, continuity of measures gives $\nu_n(E_x) \to \nu(E_x)$

and $\mu_n(E^y) \to \mu(E^y)$, so that $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are measurable as limits of measurable functions.

► **Exercise 11.14.** Let $(Z, \mathcal{H}, \alpha)$ be a measure space. Let $W \in \mathcal{H}$ and define $\beta(A) = \alpha(A \cap W)$ for all $A \in \mathcal{H}$.

Show that $(Z, \mathcal{H}, \beta)$ is a measure space.

Show that for any $f \in L^+(Z, \mathcal{H})$ we have

$$\int f d\beta = \int_W f d\alpha.$$

♣ **Solution to ex:11.14.** :(

β is a measure since $\beta(\emptyset) = \alpha(\emptyset) = 0$ and

$$\beta(\biguplus_n A_n) = \alpha(\biguplus_n (A_n \cap W)) = \sum_n \alpha(A_n \cap W) = \sum_n \beta(A_n).$$

If $f = \mathbf{1}_A$ then

$$\int f d\beta = \beta(A) = \alpha(A \cap W) = \int_W \mathbf{1}_A d\alpha.$$

If f is a simple function, then $\int f d\beta = \int_W f d\alpha$ by linearity.

If $f \in L^+$, then let $\varphi_n \nearrow f$ be a sequence of simple functions approximating f. Then, by monotone convergence,

$$\int f d\beta \leftarrow \int \varphi_n d\beta = \int_W \varphi_n d\alpha \to \int_W f d\alpha.$$

:) ✓

Using the above exercise, we conclude that

$$\mu \otimes \nu(E \cap X_n \times Y_n) = \mu_n \otimes \nu_n(E) = \int \nu(E_x \cap Y_n) d\mu_n = \int \mu(E^y \cap X_n) d\nu_n$$
$$= \int_{X_n} \nu(E_x \cap Y_n) d\mu = \int_{Y_n} \mu(E^y \cap X_n) d\nu.$$

Since $\nu(E_x \cap Y_n)\mathbf{1}_{X_n}(x) \nearrow \nu(E_x)$ and $\mu(E^y \cap Y_n)\mathbf{1}_{Y_n}(y) \nearrow \mu(E^y)$, we use Monotone Convergence to get that

$$\mu \otimes \nu(E) \lim_{n\to\infty} \mu \otimes \nu(E \cap X_n \times Y_n) = \int \nu(E_x) d\mu = \int \mu(E^y) d\nu.$$

□

11.3. The Fubini-Tonelli Theorem

★★★ **THEOREM 11.7** (Fubini-Tonelli)**.** *Let $(X, \mathcal{F}, \mu), (Y, \mathcal{G}, \nu)$ be σ-finite measure spaces.*

- *Let $f \in L^+((X, \mathcal{F}, \mu) \otimes (Y, \mathcal{G}, \nu))$. Then the functions $x \mapsto \int f_x d\nu$ and $y \mapsto \int f^y d\mu$ are in L^+.*
- *Let $f \in L^1((X, \mathcal{F}, \mu) \otimes (Y, \mathcal{G}, \nu))$. Then the functions $x \mapsto \int f_x d\nu$ and $y \mapsto \int f^y d\mu$ are defined a.e. and in L^1.*

In both cases we have

$$\int f d(\mu \otimes \nu) = \int \left(\int f_x d\nu \right) d\mu(x) = \int \left(\int f^y d\mu \right) d\nu(y).$$

Proof. Start with $f \in L^+$. If $f = \mathbf{1}_E$ this is just the previous theorem. Since $(f+g)_x = f_x + g_x$ and $(f+g)^y = f^y + g^y$, we have the theorem for all simple functions. If $f \in L^+$ is general, let $\varphi_n \nearrow f$ be an increasing sequence of approximating simple functions. It may be easily verified that $(\varphi_n)_x \nearrow f_x$ and $(\varphi_n)^y \nearrow f^y$. So f_x, f^y are indeed in L^+. Also, $(\varphi_n)_x, (\varphi_n)^y$ are simple functions. Note that $\int (\varphi_n)_x d\nu$ and $\int (\varphi_n)^y d\mu$ are also increasing sequences of functions in L^+. So by Monotone Convergence,

$$\int f d(\mu \otimes \nu) = \lim_{n\to\infty} \int \varphi_n d(\mu \otimes \nu) = \lim_{n\to\infty} \int \int (\varphi_n)_x d\nu d\mu = \int \int f_x d\nu d\mu.$$

Similarly for

$$\int f d(\mu \otimes \nu) = \int \int f^y d\mu d\nu.$$

Now for $f \in L^1$. Write $f = f_1 - f_2 + i(f_3 - f_4)$ for $f_j \in L^+$. Note that $(f_x)_j = (f_j)_x$ and $(f^y)_j = (f_j)^y$. Since f is integrable, so are f_j for all j. Thus, the first part of the theorem tells us that

$$\int \left(\int (f_j)_x d\nu \right) d\mu = \int \left(\int (f_j)^y d\mu \right) d\nu = \int f_j d(\mu \otimes \nu) < \infty, \tag{11.1}$$

which gives that $\int (f_j)_x d\nu, \int (f_j)^y d\mu$ are a.e. finite, and integrable as functions of x and y respectively. Since $|\int f_x d\nu| \leq \sum_{j=1}^4 \int (f_j)_x d\nu$, we have that $x \mapsto \int f_x d\nu$ is an L^1 function. Similarly $y \mapsto \int f^y d\mu$ is an L^1 function. We also have by summing (11.1) that

$$\int \left(\int f_x d\nu \right) d\mu = \int \left(\int f^y d\mu \right) d\nu = \int f d(\mu \otimes \nu).$$

□

▶ **Exercise 11.15.** [Folland, p.69, ex.51] Let $(X, \mathcal{F}, \mu), (Y, \mathcal{G}, \nu)$ be measure spaces that are not necessarily σ-finite. Show that:

- If $f : X \to \mathbb{C}$ is $\mathcal{F}$-measurable and $g : Y \to \mathbb{C}$ is $\mathcal{G}$-measurable then $h(x, y) := f(x)g(y)$ is $\mathcal{F} \otimes \mathcal{G}$-measurable.
- If $f \in L^1(X, \mathcal{F}, \mu)$ and $g \in L^1(Y, \mathcal{G}, \nu)$ then $h \in L^1(X \times Y, \mathcal{F} \otimes \mathcal{G}, \mu \otimes \nu)$ and

$$\int h d(\mu \otimes \nu) = \int f d\mu \cdot \int g d\nu.$$

♣ **Solution to ex:11.15.** :(

Note that he function $\phi : \mathbb{C}^2 \to \mathbb{C}$ defined by $\phi(z, w) = zw$ is continuous and thus measurable. Also, the function $H(x, y) := (f(x), g(y))$ from $X \times Y$ to $\mathbb{C}^2$ is $\mathcal{F} \otimes \mathcal{G}$-measurable since the Borel σ-algebra on $\mathbb{C}^2$ is generated by sets of the form $B_1 \times B_2$ where B_1, B_2 are Borel in $\mathbb{C}$, and for any such set $B_1 \times B_2$,

$$H^{-1}(B_1 \times B_2) = \{(x, y) \ : \ f(x) \in B_1, g(y) \in B_2\} = f^{-1}(B_1) \times g^{-1}(B_2) \in \mathcal{F} \times \mathcal{G} \subset \mathcal{F} \otimes \mathcal{G}.$$

Hence $h(x, y) = \phi(f(x), g(y)) = \phi \circ H(x, y)$ is $\mathcal{F} \otimes \mathcal{G}$-measurable as a composition of measurable functions.

For the second assertion, suppose first that $f = \mathbf{1}_A, g = \mathbf{1}_B$. Then clearly $h = \mathbf{1}_{A \times B}$ and

$$\int h d(\mu \times \nu) = (\mu \times \nu)(A \times B) = \mu(A)\nu(B) = \int f d\mu \cdot \int g d\nu.$$

If $f = \sum_{k=1}^n a_k \mathbf{1}_{A_k}$ and $g = \sum_{j=1}^m b_j \mathbf{1}_{B_j}$ are simple functions where $(A_k)_k$ and $(B_j)_j$ are both pairwise disjoint, then

$$h = \sum_{k=1}^n \sum_{j=1}^m a_k b_j \mathbf{1}_{A_k \times B_j}$$

is a simple function and

$$\int h d(\mu \times \nu) = \int f d\mu \cdot \int g d\nu$$

follows by linearity.

If $f \geq 0, g \geq 0$ then we may choose $f_n \nearrow f, g_n \nearrow g$ where f_n, g_n are simple for all n. Since multiplication is continuous, $f_n g_n \nearrow h$. Thus, by monotone convergence,

$$\int h d(\mu \times \nu) = \int f d\mu \cdot \int g d\nu.$$

If f, g are real-valued integrable functions, then

$$\begin{aligned} h(x,y) &= (f^+(x) - f^-(x)) \cdot (g^+(y) - g^-(y)) \\ &= f^+(x)g^+(y) + f^-(x)g^-(y) - f^+(x)g^-(y) - f^-(x)g^+(y). \end{aligned}$$

This is a linear combination of 4 products of non-negative functions. Since

$$\int f^\alpha(x) g^\beta(y) d(\mu \times \nu) = \int f^\alpha d\mu \cdot \int g^\beta d\nu$$

for any choice of $\alpha, \beta \in \{+, -\}$, we have by linearity

$$\int |h| d(\mu \times \nu) \leq \sum_{\alpha,\beta \in \{+,-\}} \int f^\alpha d\mu \cdot \int g^\beta d\nu < \infty,$$

so h is integrable. Also by linearity, and by the above for non-negative functions,

$$\begin{aligned} \int h d(\mu \times \nu) &= \sum_{\alpha,\beta \in \{+,-\}} \int \alpha\beta f^\alpha(x) g^\beta(y) d(\mu \times \nu) \\ &= \sum_{\alpha,\beta \in \{+,-\}} \alpha\beta \int f^\alpha d\mu \cdot \int g^\beta d\nu = \int f d\mu \cdot \int g d\nu. \end{aligned}$$

Finally for complex valued functions in L^1 write $f = f_1 + i f_2$ and $g = g_1 + i g_2$, and note that if $h(x,y) = f(x)g(y)$ then

$$h(x,y) = f_1(x)g_1(y) - f_2(x)g_2(y) + i \cdot \left[f_1(x)g_2(y) + f_2(x)g_1(y) \right].$$

So the real part of h and imaginary part of h fall into the category of the previous assertion, and so satisfy that they are in L^1 (because f_1, f_2, g_1, g_2 are all in L^1) and by linearity

$$\begin{aligned}\int hd(\mu\times\nu) &= \int f_1 d\mu\cdot\int g_1 d\nu - \int f_2 d\mu\cdot\int g_2 d\nu + i\cdot\int f_1 d\mu\cdot\int g_2 d\nu + i\cdot\int f_2 d\mu\cdot\int g_1 d\nu\\ &= \int f d\mu\cdot\int g d\nu.\end{aligned}$$

:) ✓

► **Exercise 11.16.** Let μ be the counting measure on $([0,1],\mathcal{B}([0,1]))$ (for any Borel set A, $\mu(A)=|A|$). Let λ be Lebesgue measure. Let $D=\left\{(x,x)\in[0,1]^2\right\}$. Compute $\int\left(\int \mathbf{1}_D(x,y)d\mu(x)\right)d\lambda(y), \int\left(\int \mathbf{1}_D(x,y)d\lambda(y)\right)d\mu(x)$, and show that they are not equal.

Use the fact that $\mu\otimes\lambda$ can be defined as an extension (although non-unique) of the pre-measure on boxes to compute $\int \mathbf{1}_D d(\mu\otimes\lambda)$ and show this is also not equal to the other two integrals.

♣ **Solution to ex:11.16.** :(

For any $y\in[0,1]$,

$$\int \mathbf{1}_D(x,y)d\lambda(y)=\lambda(\{x\})=0,$$

and

$$\int \mathbf{1}_D(x,y)d\mu(x)=\mu(\{y\})=1.$$

Thus,

$$\int\left(\int \mathbf{1}_D(x,y)d\mu(x)\right)d\lambda(y)=1 \qquad \text{and} \qquad \int\left(\int \mathbf{1}_D(x,y)d\lambda(y)\right)d\mu(x)=0.$$

Also, $\int \mathbf{1}_D d(\mu\otimes\lambda)=(\mu\otimes\lambda)^*(D)$, where $(\mu\otimes\lambda)^*$ is the outer measure induced by the pre-measure on the algebra of finite disjoint unions of boxes. This pre-measure is defined by $(\mu\otimes\lambda)(A\times B)=\mu(A)\lambda(B)$. Thus,

$$(\mu\otimes\lambda)^*(D)=\inf\left\{\sum_n \mu(A_n)\lambda(B_n)\ :\ D\subset\bigcup_n A_n\times B_n\right\}.$$

If $D \subset \bigcup_n A_n \times B_n$ then for any $x \in [0,1]$ we have that there exists some n for which $(x,x) \in A_n \times B_n$, so $x \in B_n \cap A_n$. Thus, $[0,1] \subset \bigcup_n (B_n \cap A_n)$, so $\sum_n \lambda(B_n \cap A_n) \geq \lambda([0,1]) = 1$. So there must exist k such that $|B_k \cap A_k|$ has $\lambda(B_k \cap A_k) > 0$, and specifically $B_k \cap A_k$ is infinite. Now, for this k we have that $\mu(A_k) \geq \mu(A_k \cap B_k) = \infty$. Hence, for any $D \subset \bigcup_n A_n \times B_n$ there exists k such that

$$\sum_n \mu(A_n)\lambda(B_n) \geq \mu(A_k)\lambda(B_k) = \infty.$$

Thus, $(\mu \otimes \lambda)^*(D) = \infty$. :) ✓

► **Exercise 11.17.** [Folland p.69] Let X be a linearly ordered set that is uncountable, but for any $x \in X$ the set $\{y \in X \ : \ y < x\}$ is countable. Let $\mathcal{F}$ be the countable-co-countable σ-algebra on X; *i.e.* $A \in \mathcal{F}$ if A is countable or if A^c is countable. Define $E = \{(x,y) \in X \times X \ : \ x < y\}$.

Show that E_x, E^y are measurable for all $x, y \in X$.

For $A \in \mathcal{F}$ define $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A^c is countable. Show that μ is a measure on $(X, \mathcal{F})$.

Show that $\int \left(\int \mathbf{1}_E(x,y) d\mu(x) \right) d\mu(y), \int \left(\int \mathbf{1}_E(x,y) d\mu(y) \right) d\mu(x)$ exist and are not equal.

♣ **Solution to ex:11.17.** :(

For any $x \in X$ let $C_x = \{y \ : \ y < x\}$. By assumption C_x is countable for all x and $(C_x)^c$ is uncountable for all x.

For any $x, y \in X$ we have $E_x = \{y \ : \ y > x\} = (C_x \uplus \{x\})^c$, so $(E_x)^c$ is countable, and thus E_x is measurable and $\mu(E_x) = 1$.

Also, $E^y = \{x \ : \ x < y\} = C_y$ which is countable, so E^y is measurable and $\mu(E^y) = 0$.

μ is easily verified to be a measure.

Finally,

$$\int \left(\int \mathbf{1}_E(x,y) d\mu(x) \right) d\mu(y) = \int \mu(E^y) d\mu(y) = 0,$$

$$\int\left(\int \mathbf{1}_E(x,y)d\mu(y)\right)d\mu(x) = \int \mu(E_x)d\mu(x) = \mu(X) = 1.$$

:) ✓

► **Exercise 11.18.** [Folland p.69] Consider $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ where μ is the counting measure. Define

$$f(n,k) = \begin{cases} 1 & n = k \\ -1 & n = k+1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\int |f| d(\mu \otimes \mu) = \infty$ and that $\int\left(\int f(x,y)d\mu(x)\right)d\mu(y), \int\left(\int f(x,y)d\mu(y)\right)d\mu(x)$ exist but are not equal.

♣ **Solution to ex:11.18.** :(

$f(n,k) = \mathbf{1}_{\{n=k\}} - \mathbf{1}_{\{n=k+1\}}$.

$|f(n,k)| = \mathbf{1}_{\{n=k\}} + \mathbf{1}_{\{n=k+1\}}$ so

$$\int |f| d(\mu \otimes \mu) = \sum_{(n,k)\in\mathbb{N}^2} (\mathbf{1}_{\{n=k\}} + \mathbf{1}_{\{n=k+1\}}) = \#\{(n,n),(n+1,n) \ : \ n \in \mathbb{N}\} = \infty.$$

For any $k \in \mathbb{N}$,

$$\int f(n,k)d\mu(n) = \sum_n \mathbf{1}_{\{n=k\}} - \mathbf{1}_{\{n=k+1\}} = 0.$$

However, for any $n \in \mathbb{N}$,

$$\int f(n,k)d\mu(k) = \sum_k \mathbf{1}_{\{n=k\}} - \mathbf{1}_{\{n=k+1\}} = \begin{cases} 0 & n > 0 \\ 1 & n = 0. \end{cases}$$

So

$$\int\int f(n,k)d\mu(n)d\mu(k) = 0 \qquad \text{and} \qquad \int\int f(n,k)d\mu(k)d\mu(n) = \int \mathbf{1}_{\{0\}}(n)d\mu(n) = 1.$$

:) ✓

▶ **Exercise 11.19.** [Folland p.69] Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space. Let $f \in L^+(X, \mathcal{F}, \mu)$. Set $\Gamma_f := \{(x, y) \in X \times [0, \infty] \ : \ y \leq f(x)\}$.

Show that $\Gamma_f \in \mathcal{F} \otimes \mathcal{B}([0, \infty])$. Show that $(\mu \otimes \lambda)(\Gamma_f) = \int f d\mu$.

♣ **Solution to ex:11.19.** :(

The functions $\varphi(x, y) = f(x) - y$ is $\mathcal{F} \otimes \mathcal{B}([0, \infty])$-measurable, as a composition of the continuous function subtraction on the measurable function $(x, y) \mapsto (f(x), y)$. Thus,

$$\Gamma_f = \{(x, y) \ : \ f(x) - y \geq 0\} = \varphi^{-1}([0, \infty]) \in \mathcal{F} \otimes \mathcal{B}([0, \infty]).$$

Note that

$$(\Gamma_f)_x = \{y \ : \ y \leq f(x)\} = \mathbf{1}_{[0, f(x)]}.$$

Now using Fubini-Tonelli,

$$(\mu \otimes \lambda)(\Gamma_f) = \int \int (\Gamma_f)_x(y) d\lambda(y) d\mu(x) = \int \lambda([0, f(x)]) d\mu(x) = \int f(x) d\mu(x).$$

:) ✓

Number of exercises in lecture: 19

Total number of exercises until here: 125

Measure Theory

Ariel Yadin

Lecture 12: Change of variables

12.1. Linear transformations of Lebesgue measure

Recall that we showed that if L is an invertible linear transformation on $\mathbb{R}^d$, for any $A \subset \mathbb{R}^d$ that is Lebesgue measurable, we have that $L(A)$ is also Lebesgue measurable and $\lambda(L(A)) = |\det L|\lambda(A)$. The proof of this fact was not so simple and kind of technical.

We now give a more general statement, and also a shorter proof using the tools we have developed.

●●● **Theorem 12.1.** *Let L be an invertible linear transformation of $\mathbb{R}^d$ and let $f : \mathbb{R}^d \to \mathbb{C}$. If f is Lebesgue measurable then so is $f \circ L$. Also, if $f \geq 0$ or if $f \in L^1$, then so is $f \circ L$ and*

$$\int f d\lambda = |\det L| \int f \circ L d\lambda.$$

Specifically, if $A \subset \mathbb{R}^d$ is Lebesgue measurable then $L(A)$ is Lebesgue measurable and $\lambda(L(A)) = |\det L|\lambda(A)$.

Proof. The main idea is as in the first proof we gave for Lebesgue measure of sets.

First suppose that f is Borel. Then $f \circ L$ is also because L is continuous.

If the theorem holds for linear maps L, M then it also holds for $L \circ M$ as:

$$\int f = |\det L| \int f \circ L = |\det L| \cdot |\det M| \int (f \circ L) \circ M = |\det L \circ M| \int f \circ L \circ M.$$

Thus, we want to decompose linear transformations into elementary types that we know how to deal with.

Every invertible linear transformation can be written as composition of the following three types: M which is multiplication of coordinate j by a non-zero scalar α, A which is adding coordinate k to coordinate j, and S which is swapping coordinates k and j. So we only need to prove the theorem for M, A, S.

Now $\det M = \alpha, \det A = 1, \det S = -1$. Thus, using the Fubini-Tonneli Theorem, we can integrate the coordinates in any order, so

$$\int f \circ M(x_1, \ldots, x_d) d\lambda^d = \int f(x_1, \ldots, \alpha x_j, \ldots, x_d) d\lambda^d = \int \int g_x(\alpha x_j) d\lambda(x_j) d\lambda^{d-1}(x),$$

where $x = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d)$ and $g_x(x_j) = f(x_1, \ldots, x_d)$. The statement

$$\int g(x) d\lambda(x) = |\alpha| \cdot \int g(\alpha x) d\lambda(x),$$

is just the $d = 1$ case, which we will prove shortly. Given the $d = 1$ case we have

$$\begin{aligned} |\alpha| \int f \circ M d\lambda^d &= \int |\alpha| \int g_x(\alpha x_j) d\lambda(x_j) d\lambda^{d-1}(x) \\ &= \int \int g_x(x_j) d\lambda(x_j) d\lambda^{d-1}(x) = \int f d\lambda^d. \end{aligned}$$

Similarly for A and S: Given the $d = 1$ case we have

$$\begin{aligned} \int f \circ A d\lambda^d &= \int \int g_x(x_j + x_k) d\lambda(x_j) d\lambda^{d-1}(x) \\ &= \int \int g_x(x_j) d\lambda(x_j) d\lambda^{d-1}(x) = \int f d\lambda^d. \end{aligned}$$

For S we get by changing the order of integration (Fubini-Tonelli),

$$\begin{aligned} &\int f \circ S(x_1, \ldots, x_j, \ldots, x_k, \ldots, x_d) d\lambda^d \\ &\quad = \int f(x_1, \ldots, x_k, \ldots, x_j, \ldots, x_d) d\lambda(x_1) \cdots d\lambda(x_j) \cdots d\lambda(x_k) \cdots d\lambda(x_d) = \int f d\lambda^d. \end{aligned}$$

So we are left with verifying the $d = 1$ cases of M and A:

$$\int g d\lambda = |\alpha| \cdot \int g(\alpha x) d\lambda(x) \qquad \text{and} \qquad \int g d\lambda = \int g(x + a) d\lambda(x).$$

If $g = \mathbf{1}_A$ then this is just $|\alpha|\lambda(\alpha^{-1}A) = \lambda(A)$ and $\lambda(A + a) = \lambda(A)$. By additivity this extends to simple functions. Monotone Convergence gives this for non-negative functions. Decomposing g into real and imaginary parts and those into positive and negative parts gives the general case.

This completes the theorem for Borel functions.

If f is Lebesgue then there exists a Borel function g and a Borel set N such that $\lambda(N) = 0$ and $g\mathbf{1}_{N^c} = f\mathbf{1}_{N^c}$. We have essentially already proved this in an exercise, but the next exercise also reproves this. Since

$$(f\circ L)(x)\cdot(\mathbf{1}_{N^c}\circ L^{-1})(x)=f(L(x))\cdot\mathbf{1}_{N^c}(L(x))=g(L(x))\cdot\mathbf{1}_{N^c}(L(x))=(g\circ L)(x)\cdot(\mathbf{1}_{N^c}\circ L^{-1})(x),$$

we have that $(f\circ L)\cdot(\mathbf{1}_{N^c}\circ L^{-1})$ is Borel. So $f\circ L$ is Lebesgue. Moreover, because $\lambda(N)=\lambda(L^{-1}(N))=0$,

$$\int f d\lambda=\int g d\lambda=|\det L|\int(g\circ L)d\lambda=|\det L|\int(f\circ L)d\lambda.$$

□

► **Exercise 12.1.** Show that a function f is Lebesgue if and only if there exists a Borel function g and a Borel set N such that $\lambda(N)=0$ and $g\mathbf{1}_{N^c}=f\mathbf{1}_{N^c}$.

♣ **Solution to ex:12.1.** :(

If there exists a Borel function g and a Borel set N such that $\lambda(N)=0$ and $g\mathbf{1}_{N^c}=f\mathbf{1}_{N^c}$, then for any Borel set B in the image, $f(x)\in B$ if and only if $g(x)\in B, x\notin N$ or $f(x)\in B, x\in N$. Thus $f^{-1}(B)=g^{-1}(B)\cap N^c\uplus f^{-1}(B)\cap N$. Since $f^{-1}(B)\cap N\subset N$ which has measure 0, we get that $f^{-1}(B)\cap N$ is a Lebesgue null set, and thus Lebesgue measurable. Since $g^{-1}(B)\cap N^c$ is Borel, we have that $f^{-1}(B)$ is Lebesgue. So in this case f is a Lebesgue function.

For the other direction, if f is Lebesgue, consider the case $f=\mathbf{1}_A$ for a Lebesgue set A. Since $A=B\cup F$ for a Borel set B and a null set F (not necessarily Borel). Since $\lambda^*(F)=0$, there exists a Borel set $N\supset F$ such that $\lambda(N)=0$. Thus, $\mathbf{1}_A\mathbf{1}_{N^c}=\mathbf{1}_B\mathbf{1}_{N^c}$, and we are done taking $g=\mathbf{1}_B$.

By additivity this extends to simple functions: If $f=\sum_{j=1}^n a_j\mathbf{1}_{A_j}$ for Lebesgue $(A_j)_j$, then for every j there exist a Borel set B_j and a Borel null set N_j such that $\mathbf{1}_{A_j}\mathbf{1}_{(N_j)^c}=\mathbf{1}_{B_j}\mathbf{1}_{(N_j)^c}$. Taking $N=\bigcup_{j=1}^n N_j$ and $g=\sum_{j=1}^n a_j\mathbf{1}_{B_j}$, we have that g is Borel and $f\mathbf{1}_{N^c}=g\mathbf{1}_{N^c}$.

Now if $f\geq 0$, let $\varphi_k\nearrow f$ be a monotone sequence of Lebesgue simple functions approximating f. For every k there exists a Borel null set N_k and a Borel (simple) function g_k such that $\varphi_k \mathbf{1}_{(N_k)^c} = g_k \mathbf{1}_{(N_k)^c}$. Taking $N = \bigcup_k N_k$ which is a Borel null set, we have that $(g_k \mathbf{1}_{N^c} = \varphi_k \mathbf{1}_{N^c})_k$ form an increasing sequence of Borel functions such that $g_k \mathbf{1}_{N^c} \nearrow f \mathbf{1}_{N^c}$. So $g = f \mathbf{1}_{N^c}$ is a Borel function.

Finally, if f is a general Lebesgue function, write $f = f_1 - f_2 + i(f_3 - f_4)$ where f_j are all non-negative Lebesgue functions. Then, there exist Borel functions g_j and Borel null sets N_j, $j = 1, 2, 3, 4$, such that $g_j \mathbf{1}_{(N_j)^c} = f_j \mathbf{1}_{(N_j)^c}$. Taking $N = N_1 \cup N_2 \cup N_3 \cup N_4$ which is a Borel null set, and $g = g_1 - g_2 + i(g_3 - g_4)$ we have that $g \mathbf{1}_{N^c} = f \mathbf{1}_{N^c}$. :) ✓

► **Exercise 12.2.** Complete the details of the previous theorem; that is, show that

$$\int g d\lambda = |\alpha| \cdot \int g(\alpha x) d\lambda(x) \qquad \text{and} \qquad \int g d\lambda = \int g(x + a) d\lambda(x).$$

► **Exercise 12.3.** Show that if λ is Lebesgue measure on $\mathbb{R}^d$ and U is a unitary operator on $\mathbb{R}^d$ then $\lambda(U(A)) = \lambda(A)$.

Specifically, Lebesgue measure is invariant under rotations.

12.2. Change of variables formula

Let $U \subset \mathbb{R}^d$ be an open set. Suppose that $\Psi : \mathbb{R}^d \to \mathbb{R}^d$ is a map such that for $\Psi = (\psi_1, \ldots, \psi_d)$ all the maps ψ_i have continuous partial derivatives $\frac{\partial \psi_i}{\partial x_j}$ on U, in all coordinates x_j. (*i.e.* ψ_i are all C^1 in U). For any $x \in \mathbb{R}^d$ define a matrix $D\Psi(x)$ by $(D\Psi(x))_{i,j} = \frac{\partial \psi_i}{\partial x_j}(x)$.

Ψ is called a C^1 **diffeomorphism** (on U) if Ψ is 1-1 and $D\Psi(x)$ is invertible for all $x \in U$. In this case by the inverse function theorem, $\Psi^{-1} : \Psi(U) \to U$ is also a C^1 diffeomorphism, and $D\Psi^{-1}(y) = (D\Psi(\Psi^{-1}(y)))^{-1}$.

► **Exercise 12.4.** Show that if L is an invertible linear map then it is a diffeomorphism with $DL(x) = L(x)$ for all $x \in \mathbb{R}^d$.

Also, show that for any diffeomorphism Ψ, $D(L \circ \Psi)(x) = L(D\Psi(x))$.

••• **Theorem 12.2.** *Let $\Psi : U \to \mathbb{R}^d$ be a C^1 diffeomorphism for an open set $U \subset \mathbb{R}^d$. If $f : \Psi(U) \to \mathbb{C}$ is Lebesgue, then $f \circ \Psi$ is Lebesgue on U. If in addition $f \geq 0$ or f is integrable, then*

$$\int_{\Psi(U)} f d\lambda = \int_U (f \circ \Psi)(x) \cdot |\det D\Psi(x)| d\lambda(x).$$

Proof. Start with Borel f. Because Ψ is continuous, $f \circ \Psi$ is Borel.

We consider the sup-norm $||x|| = \max_j |x_j|$ and for a matrix $M \in \mathrm{GL}_d(\mathbb{R})$, the operator norm $||M|| = \max_i \sum_j |M_{i,j}|$. So $||Mx|| \leq ||M||||x||$.

For $a \in \mathbb{R}^d$ and $\varepsilon > 0$ write $Q(a, \varepsilon) = \{x \ : \ ||x - a|| \leq \varepsilon\}$. Note that $Q(a, \varepsilon)$ is a box.

Step I. We show that for $Q = Q(a, \varepsilon)$,

$$\lambda(\Psi(Q)) \leq \int_Q |\det D\Psi(x)| d\lambda(x).$$

Since ψ_j are all C^1 functions the mean value theorem tells us that

$$\psi_i(x) - \psi_i(a) = \sum_{j=1}^{d} (x_j - a_j) \tfrac{\partial \psi_i}{\partial x_j}(y),$$

for some y on the line segment between x and a. Thus, for any $x \in Q = Q(a, \varepsilon)$,

$$||\Psi(x) - \Psi(a)|| \leq \varepsilon \cdot \sup_{y \in Q} \max_i \sum_j (D\Psi(y))_{i,j} = \varepsilon \cdot \sup_{y \in Q} ||D\Psi(y)||.$$

Thus,

$$\Psi(Q) \subset Q(a, \varepsilon \cdot \sup_{y \in Q} ||D\Psi(y)||).$$

By the scaling of Lebesgue measure

$$\lambda(\Psi(Q)) \leq \lambda(Q(a, \varepsilon \cdot \sup_{y\in Q} ||D\Psi(y)||)) = \lambda(\sup_{y\in Q} ||D\Psi(y)|| \cdot Q) = (\sup_{y\in Q} ||D\Psi(y)||)^d \cdot \lambda(Q).$$

For any invertible linear map $L \in \mathrm{GL}_d(\mathbb{R})$,

$$\lambda(\Psi(Q)) = |\det L| \lambda(L^{-1} \circ \Psi(Q)) \leq |\det L| \cdot (\sup_{y\in Q} ||L^{-1} D\Psi(y)||)^d \cdot \lambda(Q).$$

Now, $y \mapsto D\Psi(y)$ is a continuous function, and so uniformly continuous on the compact set Q. Thus, for any $\eta > 0$ there exists $k > 0$ such that for all $||z - y|| \leq \frac{1}{k}, z, y \in Q$ we have $||(D\Psi(z))^{-1} D\Psi(y)||^d \leq 1 + \eta$. Write $Q = \bigcup_{j=1}^{n_k} Q_{k,j}$ where $Q_{k,j} = Q(x_{k,j}, \frac{1}{k})$ and all have disjoint interiors. Then,

$$\begin{aligned}
\lambda(\Psi(Q)) &= \sum_{j=1}^{n_k} \lambda(\Psi(Q_{k,j})) \\
&\leq \sum_{j=1}^{n_k} |\det D\Psi(x_{k,j})| \cdot (\sup_{y\in Q_j} ||(D\Psi(x_{k,j}))^{-1} D\Psi(y)||)^d \cdot \lambda(Q_{k,j}) \\
&\leq (1+\eta) \int \sum_{j=1}^{n_k} |\det D\Psi(x_{k,j})| \mathbf{1}_{Q_{k,j}}(x) d\lambda(x).
\end{aligned}$$

This holds for any $Q = \bigcup_{j=1}^{n_k} Q_{k,j}$ where $Q_{k,j} Q(x_{k,j}, \frac{1}{k})$, and all have disjoint interiors.

The integrand $\sum_{j=1}^{n_k} |\det D\Psi(x_{k,j})| \mathbf{1}_{Q_{k,j}}(x)$ is continuous, and on Q tends uniformly to $|\det D\Psi(x)| \mathbf{1}_Q(x)$ as $k \to \infty$. By Dominated Convergence that

$$\lambda(\Psi(Q)) \leq (1+\eta) \lim_{k\to\infty} \sum_{j=1}^{n_k} \int_{Q_{k,j}} |\det D\Psi(x_{k,j})| d\lambda(x) = (1+\eta) \int_Q |\det D\Psi(x)| d\lambda(x).$$

Thus, sending $\eta \to 0$,

$$\lambda(\Psi(Q)) \leq \int_Q |\det D\Psi(x)| d\lambda(x).$$

Step II. We show that for any open set U,

$$\lambda(\Psi(U)) \leq \int_U |\det D\Psi(x)| d\lambda(x).$$

Indeed, if U is an open set, we may write $U = \bigcup_n Q_n$ where Q_n are almost disjoint boxes (*i.e.* have disjoint interiors). Thus,

$$\lambda(\Psi(U)) \leq \sum_n \lambda(\Psi(Q_n)) \leq \sum_n \int_{Q_n} |\det D\Psi(x)| d\lambda(x) = \int_U |\det D\Psi(x)| d\lambda(x).$$

Step III. We show that for any Borel set A,

$$\lambda(\Psi(A)) \le \int_A |\det D\Psi(x)| d\lambda(x).$$

Indeed, for a Borel set A with $\lambda(A) < \infty$, there exist open sets $(U_n)_n$ such that $A \subset U_n$ for all n and $\lambda(U_n \setminus A) \to 0$. By replacing U_n with $\bigcap_{j=1}^n U_j$ we may assume that $(U_n)_n$ is a decreasing sequence. Thus, continuity of measure gives that for $U = \bigcap_n U_n$ we have $A \subset U$ and $\lambda(U \setminus A) = 0$. So $\mathbf{1}_{U_n} \to \mathbf{1}_A$ a.e., and the Dominated Convergence Theorem now gives that

$$\begin{aligned}\lambda(\Psi(A)) \le \lambda(\Psi(U)) &= \lim_{n\to\infty} \lambda(\Psi(U_n)) \\ &\le \lim_{n\to\infty} \int_{U_n} |\det D\Psi(x)| d\lambda(x) = \int_A |\det D\Psi(x)| d\lambda(x).\end{aligned}$$

Now if A is a general Borel set then $A = \biguplus_n A_n$ where $(A_n)_n$ all have finite measure (this is just σ-finiteness). So

$$\lambda(\Psi(A)) \le \sum_n \lambda(\Psi(A_n)) \le \sum_n \int_{A_n} |\det D\Psi(x)| d\lambda(x) = \int_A |\det D\Psi(x)| d\lambda(x).$$

Step IV. If $f = \sum_{j=1}^n a_j \mathbf{1}_{A_j}$ is a Borel simple function defined on $\Psi(U)$,

$$\begin{aligned}\int_{\Psi(U)} f d\lambda &= \sum_{j=1}^n a_j \lambda(A_j) \le \sum_{j=1}^n a_j \int \mathbf{1}_{\Psi^{-1}(A_j)}(x) \cdot |\det D\Psi(x)| d\lambda(x) \\ &= \int_U |\det D\Psi(x)| \cdot \sum_{j=1}^n a_j \mathbf{1}_{A_j}(\Psi(x)) d\lambda(x) = \int_U (f \circ \Psi)(x) \cdot |\det D\Psi(x)| d\lambda(x).\end{aligned}$$

Taking limits with the monotone convergence theorem gives that for any Borel $f \ge 0$ defined on $\Psi(U)$,

$$\int_{\Psi(U)} f d\lambda \le \int_U (f \circ \Psi)(x) \cdot |\det D\Psi(x)| d\lambda(x).$$

Replacing f with the function $(f \circ \Psi)(x) \cdot |\det D\Psi(x)|$ which is defined on U we have

$$\int_U (f \circ \Psi)(x) \cdot |\det D\Psi(x)| d\lambda(x) \le \int_{\Psi(U)} (f \circ \Psi \circ \Psi^{-1})(x) \cdot |\det D\Psi(\Psi^{-1}(x))| \cdot |\det D\Psi^{-1}(x)| d\lambda(x).$$

Since $D\Psi(\Psi^{-1}(x)) = (D\Psi^{-1}(x))^{-1}$, we have that $|\det D\Psi(\Psi^{-1}(x))| \cdot |\det D\Psi^{-1}(x)| = 1$, giving

$$\int_U (f \circ \Psi)(x) \cdot |\det D\Psi(x)| d\lambda(x) \le \int_{\Psi(U)} f d\lambda \le \int_U (f \circ \Psi)(x) \cdot |\det D\Psi(x)| d\lambda(x).$$

This establishes the theorem for all non-negative Borel f.

To get general integrable Borel f just decompose into real, imaginary, positive and negative parts. To get Lebesgue f find a Borel function that is a.e. identical to f. We omit the details. $\square$

Example 12.3. Let us compute $I := \int_{-\infty}^{\infty} e^{-x^2} dx$. Consider

$$I^2 = \int \int e^{-x^2} e^{-y^2} dx dy = \int_{\mathbb{R}^2} f(x,y) d\lambda^2(x,y),$$

where $f(x,y) = e^{-(x^2+y^2)}$, by Fubini.

Let $U = (0,\infty) \times (-\pi,\pi)$. Let $\Psi : U \to \mathbb{R}^2$ be $\Psi(r,\theta) = (r\cos\theta, r\sin\theta)$. So $\Psi(U) = \mathbb{R}^2 \setminus ((-\infty,0] \times \{0\})$. Since $\lambda^2((-\infty,0] \times \{0\}) = 0$ we have that $I^2 = \int_{\Psi(U)} f d\lambda^2$.

Note that

$$D\Psi(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}.$$

So $|\det D\Psi(r,\theta)| = r$.

We get using Fubini again, since $f(\Psi(r,\theta)) = e^{-r^2}$,

$$\begin{aligned} I^2 &= \int_{\Psi(U)} f d\lambda^2 = \int_U f(\Psi(r,\theta)) \cdot r d\lambda(r,\theta) \\ &= \int_{-\pi}^{\pi} \int_0^{\infty} r e^{-r^2} dr d\theta = 2\pi \cdot \int_0^{\infty} (-\tfrac{1}{2}) \tfrac{d}{dr} e^{-r^2} dr = \pi. \end{aligned}$$

So $I = \sqrt{\pi}$. △ ▽ △

Number of exercises in lecture: 4

Total number of exercises until here: 129

Measure Theory

Ariel Yadin

Lecture 13: Lebesgue-Radon-Nykodim

13.1. Signed measures

• **Definition 13.1.** Let $(X, \mathcal{F})$ be a measurable space. A **signed measure** on $(X, \mathcal{F})$ is a function $\nu : \mathcal{F} \to [-\infty, \infty]$ such that

- $\nu(\emptyset) = 0$;
- $\nu(\mathcal{F}) \subset (-\infty, \infty]$ or $\nu(\mathcal{F}) \subset [-\infty, \infty)$; that is, ν takes on at most one of the values $-\infty, \infty$ (but not both);
- If $(A_n)_n$ is a sequence of pairwise disjoint sets in $\mathcal{F}$ then $\nu(\biguplus_n A_n) = \sum_n \nu(A_n)$, and if $|\nu(\biguplus_n A_n)| < \infty$ then $\sum_n |\nu(A_n)| < \infty$.

When we say **positive measure** we stress that the measure in question is a measure that is not signed; *i.e.* in the usual sense.

We really only require signed measures in order to prove differentiation theorems.

► **Exercise 13.1.** Show that if μ_1, μ_2 are measures on $(X, \mathcal{F})$ and at least one of them is a finite measure, then $\mu_1 - \mu_2$ is a signed measure.

► **Exercise 13.2.** Let f be a measurable function $f : (X, \mathcal{F}) \to [-\infty, \infty]$. Assume that one of the integrals $\int f^+ d\mu, \int f^- d\mu$ is finite, in which case the integral $\int f d\mu$ exists.

Show that $\nu(A) = \int_A f d\mu$ is a signed measure.

► **Exercise 13.3.** Let ν be a signed measure on $(X, \mathcal{F})$. Let $(A_n)_n$ be a sequence in $\mathcal{F}$.

Show that if $(A_n)_n$ is increasing, then

$$\nu(\bigcup_n A_n) = \lim_{n\to\infty} \nu(A_n).$$

Show that if $(A_n)_n$ is decreasing and $|\nu(A_1)| < \infty$ then

$$\nu(\bigcap_n A_n) = \lim_{n\to\infty} \nu(A_n).$$

• **Definition 13.2.** Let ν be a signed measure on $(X, \mathcal{F})$. We say that $A \in \mathcal{F}$ is **positive** (respectively **negative**) if for every $B \subset A$ such that $B \in \mathcal{F}$ we have $\nu(B) \geq 0$ (respectively $\nu(B) \leq 0$).

If A is both positive and negative we say A is **null**.

► **Exercise 13.4.** Let μ be a positive measure on $(X, \mathcal{F})$. Let f be a measurable function $f : (X, \mathcal{F}) \to [-\infty, \infty]$ such that the integral $\int f d\mu$ exists. Let $\nu(A) = \int_A f d\mu$, which we know is a signed measure.

Show that $A \in \mathcal{F}$ is ν-positive (respectively negative, null) if and only if $f\mathbf{1}_A \geq 0$ a.e. (respectively $f\mathbf{1}_A \leq 0, f\mathbf{1}_A = 0$ a.e.).

♣ **Solution to ex:13.4.** :(

Let $A \in \mathcal{F}$.

If A is positive then fix $\varepsilon > 0$ and let $B = \{f\mathbf{1}_A \leq -\varepsilon\}$ which is measurable because $f\mathbf{1}_A$ is measurable. Note that $B = A \cap \{f \leq -\varepsilon\} \subset A$, so $\nu(B) \geq 0$. Also, $f\mathbf{1}_B \leq$

$(-\varepsilon)\mathbf{1}_B$, and

$$0 \leq \nu(B) = \int f\mathbf{1}_B d\mu \leq -\varepsilon \cdot \mu(B).$$

So $\mu(B) = 0$ (which also implies that $\nu(B) = 0$.)

On the other hand, if $f\mathbf{1}_A \geq 0$ a.e. then for any $\mathcal{F} \ni B \subset A$ we have that $f\mathbf{1}_B = f\mathbf{1}_A\mathbf{1}_B \geq 0$ a.e., so

$$\nu(B) = \int f\mathbf{1}_B d\mu \geq 0.$$

For the negative case, note that $A \in \mathcal{F}$ is ν-negative if and only if A is $(-\nu)$-positive, where $-\nu$ is the signed measure defined by $-\nu(B) = -\int_B f d\mu = \int_B (-f) d\mu$, we have that A is negative if and only if $(-f)\mathbf{1}_A \geq 0$ a.e., which is if and only if $f\mathbf{1}_A \leq 0$ a.e.

For the null case, $A \in \mathcal{F}$ is null if and only if it is both positive and negative, which is if and only if $0 \leq f\mathbf{1}_A \leq 0$ a.e. :) ✓

► **Exercise 13.5.** Show that if A is positive then any $B \in \mathcal{F}$ such that $B \subset A$ is also positive.

Show that a countable union of positive sets is positive.

♣ **Solution to ex:13.5.** :(

The first assertion is easy.

For the second, let $(A_n)_n$ be a sequence of positive sets and let $A = \bigcup_n A_n$. Then, $B_n = A_n \setminus \bigcup_{j=1}^{n-1} A_j \subset A_n$ is also positive. Since $A = \biguplus_n B_n$, for any measurable $C \subset A$ we have that

$$\nu(C) = \sum_n \nu(C \cap B_n) \geq 0.$$

:) ✓

••• **Theorem 13.3** (Hahn Decomposition Theorem)**.** *If ν is a signed measure on $(X, \mathcal{F})$ then there exist disjoint measurable sets $P \cap N = \emptyset$ such that $X = P \uplus N$ and P is positive and N is negative.*

This is called a **Hahn decomposition** *of X.*

If P', N' are another Hahn decomposition of X we have that $P \triangle P' = N \triangle N'$ is a null set.

Proof. By possible considering $-\nu$ instead of ν, we may assume that $\nu(A) < \infty$ for all $A \in \mathcal{F}$.

Let $m = \sup\{\nu(P) \ : \ P \text{ is positive }\}$. Let $(P_n)_n$ be a sequence of positive sets such that $\nu(P_n) \to m$. Let $P = \bigcup_n P_n$. P is positive and

$$m \geq \nu(P) = \lim_{n\to\infty} \nu(\bigcup_{j=1}^{n} P_j) \geq \lim_{n\to\infty} \nu(P_n) = m.$$

So $m = \nu(P) < \infty$.

Let $N = X \setminus P$.

For any $\mathcal{F} \ni A \subset N$, if A is positive then for any $\mathcal{F} \ni B \subset A$, also $B \uplus P$ is positive, so $\nu(P) = m \geq \nu(B \uplus P) = \nu(B) + \nu(P)$, and so $\nu(B) = 0$, which implies that A is null. That is, any positive subset of N is null.

Now let $\mathcal{F} \ni A \subset N$. If $\nu(A) > 0$ then A is not null, and since A is not positive there exists $\mathcal{F} \ni B \subset A$ such that $\nu(B) < 0$. So for $C = A \setminus B \subset A$ we have that $\nu(C) = \nu(A) - \nu(B) > \nu(A)$. So for any $A \subset N$ such that $\nu(A) > 0$ define

$$n_A := \inf\left\{n \geq 1 \ : \ \exists \ B \subset A \ , \ \nu(B) > \nu(A) + n^{-1}\right\},$$

and let $B_A \subset A$ be a set such that $\nu(B_A) > \nu(A) + \frac{1}{n_A}$.

So assume for a contradiction that N is not negative. So there exists $\mathcal{F} \ni A_0 \subset N$ such that $\nu(A_0) > 0$. Given A_k define inductively $n_{k+1} = n_{A_k}$ and $A_{k+1} = B_{A_k}$. So for all $k \geq 1$ we have that $\nu(A_k) > \nu(A_{k-1}) + \frac{1}{n_k}$ and $A_k \subset A_{k-1}$. Note that

$$\nu(A_k) > \nu(A_{k-1}) + \tfrac{1}{n_k} > \cdots > \sum_{j=1}^{k} \tfrac{1}{n_j}.$$

Since $\nu(A_0) < \infty$, for $A = \bigcap_k A_k$,

$$\infty > \nu(A) = \lim_{k\to\infty} \nu(A_k) \geq \sum_k \tfrac{1}{n_k} > 0.$$

Thus, $\sum_k \frac{1}{n_k} < \infty$ so $n_k \to \infty$. Also, $\nu(A) > 0$. Take k large enough so that $n_k > 2n_A$. Then, $B \subset A \subset A_{k-1}$. By the definition of $n_k = n_{A_{k-1}}$ we have that

$$\nu(A) + \tfrac{1}{n_A} < \nu(B) \leq \nu(A_{k-1}) + \tfrac{1}{n_k - 1} \leq \nu(A_{k-1}) + \tfrac{1}{2n_A}.$$

So $\nu(A_{k-1}) > \nu(A) + \frac{1}{2n_A}$ for all k such that $n_k > 2n_A$. Thus,

$$\nu(A) = \lim_{k \to \infty} \nu(A_k) \geq \nu(A) + \tfrac{1}{2n_A},$$

a contradiction!

Thus, N is negative.

Finally if P', N' is a Hahn decomposition, then $P \setminus P' \subset N' \cap P$, so must be both positive and negative. Similarly, $P' \setminus P \subset N \cap P'$. □

●●● **Theorem 13.4** (Jordan Decomposition Theorem)**.** *If ν is a signed measure on $(X, \mathcal{F})$ then there exist unique positive measures ν^+, ν^- such that $\nu = \nu^+ - \nu^-$ and such that $X = P \uplus N$ where $\nu^+(N) = \nu^-(P) = 0$.*

Proof. Let $X = P \uplus N$ be a Hahn decomposition. Define

$$\nu^+(A) = \nu(A \cap P) \qquad \text{and} \qquad \nu^-(A) = -\nu(A \cap N).$$

These are positive measures because P is positive and N is negative. It is immediate that $\nu = \nu^+ - \nu^-$ and that $\nu^+(N) = \nu^-(P) = 0$.

Now for uniqueness: Suppose that $\nu = \mu^+ - \mu^-$ for positive measure μ^+, μ^- and $X = P' \uplus N'$ where $\mu^+(N') = \mu^-(P') = 0$. For any $A \subset P'$ we have that $\nu(A) = \mu^+(A) \geq 0$ and for any $B \subset N'$ we have $\nu(B) = -\mu^-(B) \leq 0$. So P', N' for a Hahn decomposition, and so $P \triangle P'$ is ν-null. For any $A \in \mathcal{F}$, since $(A \cap P) \triangle (A \cap P') \subset P \triangle P'$,

$$\mu^+(A) = \mu^+(A \cap P') = \nu(A \cap P') = \nu(A \cap P) = \nu^+(A \cap P) = \nu^+(A),$$

and similarly,

$$\mu^-(A) = \mu^-(A \cap N') = -\nu(A \cap N') = -\nu(A \cap N) = \nu^-(A \cap N) = \nu^-(A).$$

□

• **Definition 13.5.** $\nu = \nu^+ - \nu^-$ is called the **Jordan decomposition** of ν. ν^+ is the positive part and ν^- is the negative part.

We also define the **total variation** or **absolute value** of ν as $|\nu| := \nu^+ + \nu^-$.

ν is called finite (respectively, σ-finite) if $|\nu$ is a finite (respectively, σ-finite) measure.

► **Exercise 13.6.** Show that A is ν-null if and only if $\nu^+(A) = \nu^-(A) = 0$, which is if and only if $|\nu|(A) = 0$.

► **Exercise 13.7.** Show that if P, N is a Hahn decomposition then $\nu^+(N) = \nu^-(P) = 0$.

► **Exercise 13.8.** Show that for all $A \in \mathcal{F}$,

$$\nu(A) = \int_A (\mathbf{1}_P - \mathbf{1}_N) d|\nu|,$$

where P, N are any Hahn decomposition.

♣ **Solution to ex:13.8.** :(

If P, N are a Hahn decomposition, then $\nu^+(N) = \nu^-(P) = 0$. So

$$\nu(A) = \nu^+(A \cap P) - \nu^-(A \cap N) = |\nu|(A \cap P) - |\nu|(A \cap N) = \int_A \mathbf{1}_P d|\nu| - \int_A \mathbf{1}_N d|\nu|.$$

We may combine the integrals into one since at least one of them is finite, because either ν^+ or ν^- is a finite measure. :) ✓

► **Exercise 13.9.** Show that $L^1(\nu^+) \cap L^1(\nu^-) = L^1(|\nu|)$.

● **Definition 13.6.** For a signed measure and $f \in L^1(|\nu|)$ we define

$$\int f d\nu = \int f d\nu^+ - \int f d\nu^-.$$

► **Exercise 13.10.** Show that for $f \in \mathrm{L}^1(|\nu|)$,

$$|\int f d\nu| \leq \int |f| d|\nu|.$$

► **Exercise 13.11.** Show that for measurable A,

$$|\nu|(A) = \sup\left\{ |\int_A f d\nu| \ : \ |f| \leq 1 \right\}.$$

► **Exercise 13.12.** Show that if $\nu = \mu - \rho$ where μ, ρ are positive measures (of which one is finite), then $\mu \geq \nu^+, \rho \geq \nu^-$.

► **Exercise 13.13.** Let ν_1, ν_2 be signed measures such that $\nu_1 < \infty, \nu_2 < \infty$.

Show that $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$.

Show that $|-\nu_1| = |\nu_1|$.

• **Definition 13.7.** Let ν, μ be signed measures on $(X, \mathcal{F})$. We say that ν, μ are **mutually singular** (or just **singular**), denoted $\nu \perp \mu$, if $X = A \uplus B$ where A is ν-null and B is μ-null.

► **Exercise 13.14.** Show that $\nu^+ \perp \nu^-$.

Show that $\nu \perp \mu$ if and only if $|\nu| \perp \mu$ if and only if $\nu^+ \perp \mu, \nu^- \perp \mu$.

• **Definition 13.8.** Let ν, μ be signed measures on $(X, \mathcal{F})$. We say that ν is **absolutely continuous** with respect to μ, denoted $\nu \ll \mu$, if every μ-null set is also a ν-null set.

► **Exercise 13.15.** Show that $\nu \ll \mu$ if and only if $|\nu| \ll |\mu|$ if and only if $\nu^+ \ll |\mu|, \nu^- \ll |\mu|$.

► **Exercise 13.16.** Show that if $\nu \ll \mu$ and $\nu \perp \mu$ then $\nu \equiv 0$.

► **Exercise 13.17.** Let $\nu(A) = \int_A f d\mu$ for some measurable f such that $f < \infty$ (but perhaps f can take the value $-\infty$). Show that $\nu \ll \mu$. Show that ν is finite if and only if f is integrable.

♣ **Solution to ex:13.17.** :(

If $\mu(A) = 0$ then $f\mathbf{1}_A = 0$ μ-a.e. So $\nu(A) = \int f\mathbf{1}_A d\mu = 0$. :) ✓

• **Proposition 13.9.** *Let ν be a finite signed measure and μ a positive measure on $(X, \mathcal{F})$. $\nu \ll \mu$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\nu(A)| < \varepsilon$ for all $A \in \mathcal{F}$ such that $\mu(A) < \delta$.*

Proof. The ε, δ condition implies that for all $A \in \mathcal{F}$ such that $\mu(A) = 0$, we have that for any $\varepsilon > 0$, $|\nu(A)| < \varepsilon$. Thus, $\mu(A) = 0$ implies $\nu(A) = 0$. So if A is a μ-null set, then for any $B \subset A$ we have $\mu(B) = 0$ so also $\nu(B) = 0$. This implies that A is a ν-null set.

For the other direction, assume that ν is a positive measure. Suppose the ε, δ condition fails. So there exists $\varepsilon > 0$ such that for any n there exists a set $A_n \in \mathcal{F}$ with $\mu(A_n) < 2^{-n}$ but $\nu(A_n) \geq \varepsilon$. Define

$$F = \limsup A_n = \bigcap_n \bigcup_{k \geq n} A_k.$$

Since $\bigcup_{k \geq n} A_k$ is a decreasing sequence, and since ν is a finite measure,

$$\nu(F) = \lim_{n \to \infty} \nu(\bigcup_{k \geq n} A_k) \geq \lim_{n \to \infty} \nu(A_n) \geq \varepsilon.$$

Also, for any n,

$$\mu(\bigcup_{k \geq n} A_k) \leq \sum_{k \geq n} \mu(A_k) \leq \sum_{k \geq n} 2^{-k} = 2^{-n+1}.$$

Thus, $\mu(F) \leq \inf_n 2^{-n+1} \leq 0$. So F is a μ-null set and $\nu(F) > 0$, which implies that it is not the case that $\nu \ll \mu$.

Now, if ν is a signed measure then $\nu \ll \mu$ if and only if $\nu^+, \nu^- \ll \mu$. So for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mu(A) < \delta$ then $\nu^+(A), \nu^-(A) < \varepsilon$. This implies that $|\nu(A)| = \nu^+(A) + \nu^-(A) < \varepsilon$. □

► **Exercise 13.18.** Show that for any integrable f, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mu(A) < \delta$ then $\left| \int_A f d\mu \right| < \varepsilon$.

We are almost ready to prove the basic differentiation theorem. A technical lemma first:

• **Lemma 13.10.** *Let ν, μ be finite positive measures on $(X, \mathcal{F})$. Either $\nu \perp \mu$ or there exist some $\varepsilon > 0$ and $A \in \mathcal{F}$ such that $\mu(A) > 0$ and A is positive for the signed measure $\nu - \varepsilon\mu$.*

Proof. Let $X = P_n \uplus N_n$ be a Hahn decomposition for the signed measure $\nu - \frac{1}{n}\mu$. Let $P = \bigcup_n P_n$ and $N = \bigcap_n N_n = P^c$. For any n we have that $N \subset N_n$ is negative for $\nu - \frac{1}{n}\mu$. That is, $0 \leq \nu(N) \leq \frac{1}{n}\mu(N)$ for all n, and so $\nu(N) = 0$.

If $\mu(P) = 0$ then $\nu \perp \mu$.

Otherwise, there exists some n for which $\mu(P_n) > 0$. Since P_n is positive for $\nu - \frac{1}{n}\mu$, we can take $\varepsilon = \frac{1}{n}$ and $A = P_n$. □

We have seen that measures of the form $A \mapsto \int_A f d\mu$ are absolutely continuous with respect to μ. The next theorem tells us that these are the only examples.

⋆⋆⋆ **THEOREM 13.11** (Lebesgue-Radon-Nikodym Theorem)**.** *Let ν be a σ-finite signed measure and let μ be a σ-finite positive measure on $(X, \mathcal{F})$. There exist unique signed measures σ, ρ such that $\sigma \perp \mu, \rho \ll \mu$, and $\nu = \sigma + \rho$. Moreover, there exists a measurable function $f : X \to [-\infty, \infty]$ such that for all $A \in \mathcal{F}$, $\rho(A) = \int_A f d\mu$ (specifically the integral exists).*

Moreover, ρ is positive if and only if f is positive. ρ is finite if and only if f is integrable. σ, ρ are positive if and only if ν is positive. σ, ρ are finite if and only if ν is finite.

✓ The decomposition $\nu = \sigma + \rho$ is called the **Lebesgue decomposition**. We use the notation $\frac{d\rho}{d\mu}$ to denote the function f given by the theorem. This function is μ-a.e. unique, and called the **Radon-Nikodym derivative** of ρ with respect to μ.

Proof. **Case I.** ν, μ are finite and positive. Define

$$M = \left\{ f : X \to [0, \infty] \ : \ \int_A f d\mu \leq \nu(A) \ \forall \ A \in \mathcal{F} \right\}.$$

$0 \in M$ so $M \neq \emptyset$. If $f, g \in M$ then for $B = \{f > g\}$ and for any $A \in \mathcal{F}$,

$$\int_A (f \vee g) d\mu = \int_{A \cap B} f d\mu + \int_{A \cap B^c} g d\mu \leq \nu(A \cap B) + \nu(A \cap B^c) = \nu(A).$$

So $f \vee g \in M$ as well.

Set $m = \sup_{f\in M} \int f d\mu \le \nu(X) < \infty$. Let $(f_n)_n \subset M$ such that $\int f_n d\mu \to m$. Let $g_n = \max\{f_1, \ldots, f_n\}$. So $(g_n)_n$ is an increasing sequence such that $g_n \nearrow f := \sup_n f_n$. Note that $g_n \in M$ for all n. So

$$m = \lim_{n\to\infty} \int f_n d\mu \le \lim_{n\to\infty} \int g_n d\mu \le m.$$

By Monotone Convergence, $m = \lim_{n\to\infty} \int g_n d\mu = \int f d\mu$. So by perhaps modifying f on a set of measure 0, we may assume that $f < \infty$. Also, for any $A \in \mathcal{F}$, by Monotone Convegence,

$$\int_A f d\mu = \lim_{n\to\infty} \int_A g_n d\mu \le \nu(A),$$

so $f \in M$.

Consider the signed measure $\sigma(A) := \nu(A) - \int_A f d\mu$. Note that σ is positive because $f \in M$. Since σ, μ are finite positive measures, if σ is not singular with respect to μ then there exist $\varepsilon > 0$ and $A \in \mathcal{F}$ such that $\mu(A) > 0$ and $\sigma(B) \ge \varepsilon\mu(B)$ for all measurable $B \subset A$. This implies that for any $C \in \mathcal{F}$,

$$\begin{aligned}\int_C (f + \varepsilon \mathbf{1}_A) d\mu &= \int_C f d\mu + \varepsilon\mu(A \cap C) \le \int_C f d\mu + \nu(A\cap C) - \int_{A\cap C} f d\mu \\ &= \int_{C\cap A^c} f d\mu + \nu(C \cap A) \le \nu(C\cap A^c) + \nu(C \cap A) = \nu(C).\end{aligned}$$

So $f + \varepsilon\mathbf{1}_A \in M$ which implies that

$$m \ge \int (f + \varepsilon\mathbf{1}_A) d\mu = \int f d\mu + \varepsilon\mu(A) > m,$$

a contradiction. So $\sigma \perp \mu$. Setting $\rho(A) = \int_A f d\mu$, since $\rho \ll \mu$, we have the decomposition for positive finite ν, μ. Uniqueness of the decomposition is proved in an exercise following.

Case II. ν, μ are σ-finite positive measures. In this case we can write $X = \biguplus_n X_n$ where $\nu(X_n) < \infty, \mu(X_n) < \infty$. For every n set $\nu_n(A) = \nu(A \cap X_n), \mu_n(A) = \mu(A\cap X_n)$. These are finite measures, so we have th? Lebesgue decomposition $\nu_n = \sigma_n + \rho_n$ where $\sigma_n \perp \mu_n$ and $\rho_n(A) = \int_A f_n d\mu_n$ for some integrable f_n. We also know that $\mu_n(X_n^c) = \nu_n(X_n^c) = 0$, so

$$\int_A f_n d\mu_n = \int_A f_n \mathbf{1}_{X_n} d\mu_n,$$

and by replacing f_n with $f_n\mathbf{1}_{X_n}$ we may assume that $f_n = 0$ off X_n. It is an exercise to show that $\int_A f_n d\mu = \int_A f_n d\mu_n$. Also, $\sigma_n(X_n^c) = \nu_n(X_n^c) - \int_{X_n^c} f_n d\mu_n = 0$.

Set $\sigma = \sum_n \sigma_n$ It is another exercise to show that $\sigma \perp \mu$. Set $f = \sum_n f_n$ and $\rho(A) = \int_A f d\mu = \sum_n \int_A f_n d\mu_n = \sum_n \rho_n(A)$. So

$$\nu(A) = \sum_n \nu_n(A) = \sum_n (\sigma_n(A) + \rho_n(A)) = \sigma(A) + \rho(A).$$

This completes the proof for σ-finite positive measures.

Case III. For σ-finite signed measure ν, write the Jordan decomposition $\nu = \nu^+ - \nu^-$. We have the Lebesgue decomposition $\nu^\pm = \sigma^\pm + \rho^\pm$ and then $\nu = (\sigma^+ - \sigma^-) + (\rho^+ - \rho^-)$ which satisfy the conditions of the theorem. □

► **Exercise 13.19.** Show that the Lebesgue decomposition is unique.

♣ **Solution to ex:13.19.** :(

Suppose that $\nu = \sigma + \rho = \sigma' + \rho'$ where $\sigma, \sigma' \perp \mu$ and $\rho, \rho' \ll \mu$. If ν is a finite measure, then $\sigma - \sigma' = \rho' - \rho$ are well defined signed measures. Since $\rho' - \rho \ll \mu$ and $\sigma - \sigma' \perp \mu$ we have that $\sigma - \sigma' = \rho' - \rho \equiv 0$.

Now, if ν is σ-finite, write $X = \biguplus X_n$ where $\nu(X_n) < \infty$. Then, define $\nu_n(A) = \nu(A \cap X_n)$. In this case, if $\nu = \sigma + \rho$ is a Lebesgue decomposition of ν with respect to μ, then for any measurable A,

$$\nu_n(A) = \nu(A \cap X_n) = \sigma(A \cap X_n) + \rho(A \cap X_n).$$

Since the signed measure $\rho_n(A) := \rho(A \cap X_n)$ is absolutely continuous with respect to μ, and since the signed measure $\sigma_n(A) := \sigma(A \cap X_n)$ is singular with respect to μ, this forms a Lebesgue decomposition of ν_n with respect to μ, which is unique because ν_n is a finite measure.

Thus, if $\nu = \sigma + \rho = \sigma' + \rho'$ where $\sigma, \sigma' \perp \mu$ and $\rho, \rho' \ll \mu$, then $\sigma(A \cap X_n) = \sigma'(A \cap X_n)$ and $\rho(A \cap X_n) = \rho(A \cap X_n)$, which implies that

$$\sigma(A) = \sum_n \sigma(A \cap X_n) = \sum_n \sigma'(A \cap X_n) = \sigma'(A),$$
$$\rho(A) = \sum_n \rho(A \cap X_n) = \sum_n \rho'(A \cap X_n) = \rho'(A).$$

:) ✓

▶ **Exercise 13.20.** Let μ be a positive measure and let Y be a measurable set of finite measure. Set $\mu_Y(A) = \mu(A \cap Y)$ for all measurable A. Let f be a measurable function such that $f = 0$ off Y.

Show that $\int_A f d\mu = \int_A f d\mu_Y$.

▶ **Exercise 13.21.** Let $(\nu_n)_n$ be a sequence of positive measures. Let μ be a positive measure.

Show that if $\nu_n \perp \mu$ for all n then $\sum_n \nu_n \perp \mu$.

Show that if $\nu_n \ll \mu$ for all n then $\sum_n \nu_n \ll \mu$.

• **Proposition 13.12** (Chain rule). *Suppose that $\nu \ll \mu \ll \rho$ for σ-finite positive measures ν, μ, ρ. If $f \in L^1(\nu)$ then $f \frac{d\nu}{d\mu} \in L^1(\mu)$ and*

$$\int f d\nu = \int f \frac{d\nu}{d\mu} d\mu.$$

Also, $\nu \ll \rho$ and $\frac{d\nu}{d\rho} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\rho}$, ρ-a.e.

Proof. If $f = \mathbf{1}_A$ then $\int f d\nu = \int f \frac{d\nu}{d\mu} d\mu$ holds by definition of the Radon-Nikodym derivative. This extends by additivity to simple functions and by Monotone convergence to non-negative functions. Taking real, imaginary, positive and negative parts proves this for all $f \in L^1(\nu)$.

For the second assertion note that for any measurable A,

$$\int_A \frac{d\nu}{d\rho} d\rho = \nu(A) = \int_A \frac{d\nu}{d\mu} d\mu = \int_A \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\rho} d\rho.$$

So the functions inside the integrals on both sides must be equal ρ-a.e. □

► **Exercise 13.22.** Show that if $\nu \ll \mu$ and $\mu \ll \nu$ then $\frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\nu} = 1$ for ν-a.e. and μ-a.e.

► **Exercise 13.23.** Let $\nu_j \ll \mu$ for a positive σ-finite μ and signed σ-finite ν_j, $j = 1, 2$. Show that $\frac{d(\nu_1+\nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}$.

► **Exercise 13.24.** Let $\nu_j \ll \mu_j$ be σ-finite measures on $(X_j, \mathcal{F}_j)$ for $j = 1, 2$. Then $\nu_1 \otimes \nu_2 \ll \mu_1 \otimes \mu_2$ and for a.e. $(x, y) \in X_1 \times X_2$,

$$\frac{d(\nu_1 \otimes \nu_2)}{d(\mu_1 \otimes \mu_2)}(x, y) = \frac{d\nu_1}{d\mu_1}(x) \cdot \frac{d\nu_2}{d\mu_2}(y).$$

► **Exercise 13.25.** Let μ be the counting measure on $([0, 1], \mathcal{B}([0, 1]))$ and λ Lebesgue measure.

Show that $\lambda \ll \mu$ but $\frac{d\lambda}{d\mu}$ does not exist. (What fails in the Radon-Nikodym Theorem?)

Show μ has no Lebesgue decomposition with respect to λ.

♣ **Solution to ex:13.25.** :(

Note that $\mu(A) = 0$ if and only if $A = \emptyset$. So $\mu(A) = 0$ implies $A = \emptyset$ which implies that $\lambda(A) = 0$. That is, $\lambda \ll \mu$.

Assume for a contradiction $\frac{d\lambda}{d\mu}$ exists. Then, for any $x \in [0, 1]$, since $\{x\}$ is Borel,

$$0 = \lambda(\{x\}) = \int \tfrac{d\lambda}{d\mu} \mathbf{1}_{\{x\}} d\mu = \tfrac{d\lambda}{d\mu}(x)\mu(\{x\}) = \tfrac{d\lambda}{d\mu}(x).$$

So $\frac{d\lambda}{d\mu} \equiv 0$ which is impossible.

The Radon-Nikodym theorem requires σ-finiteness, and μ is not σ-finite.

Now, assume that $\mu = \sigma + \rho$ where $\sigma \perp \lambda$ and $\rho \ll \lambda$. Note that for any Borel set A, if $A \neq \emptyset$ and $x \in A$ then $\sigma(A) \geq \sigma(\{x\}) = \mu(\{x\}) - \rho(\{x\}) = 1$ because $\lambda(\{x\}) = 0$. However this implies that $\sigma(A) > 0$ for any non-empty Borel set A. Since $\sigma \perp \lambda$ there exist Borel sets $[0, 1] = A \uplus B$ such that $\sigma(A) = 0$ and $\lambda(B) = 0$. But then $A = \emptyset$ and $B = [0, 1]$ which contradicts $\lambda(B) = 0$. :) ✓

► **Exercise 13.26.** Let μ, ν be σ-finite measures on $(X, \mathcal{F})$ such that $\nu \ll \mu$. Let $\lambda = \nu + \mu$.

(a) Show that $\nu \ll \lambda$.

(b) Show that $\lambda \ll \mu$.

(c) Show that $0 \leq \frac{d\nu}{d\lambda}(x) < 1$ for μ-a.e. $x \in X$.

(d) Show that μ-a.e.

$$\frac{d\nu}{d\mu} = \frac{\frac{d\nu}{d\lambda}}{1 - \frac{d\nu}{d\lambda}}.$$

♣ **Solution to ex:13.26.** :(

(a) If $A \in \mathcal{F}$ is such that $\nu(A) > 0$ then $\mu(A) > 0$, because $\nu \ll \mu$. So $\lambda(A) = \mu(A) + \nu(A) > 0$. Thus, if $\lambda(A) = 0$ then $\nu(A) = 0$. Since this holds for all $A \in \mathcal{F}$ we get that $\nu \ll \lambda$.

(b) If $A \in \mathcal{F}$ is such that $\mu(A) = 0$ then $\nu(A) = 0$ because $\nu \ll \mu$. Thus, $\lambda(A) = \mu(A) + \nu(A) = 0$. Since this holds for all $A \in \mathcal{F}$ we have that $\lambda \ll \mu$.

(c) Because λ, ν are positive measure we have that $\frac{d\nu}{d\lambda} \geq 0$. Let $A = \left\{ \frac{d\nu}{d\lambda} \geq 1 \right\}$. Then

$$\nu(A) = \int_A \frac{d\nu}{d\lambda} d\lambda \geq \lambda(A) = \nu(A) + \mu(A).$$

So $\mu(A) = 0$.

(d) Since $\frac{d\lambda}{d\mu} = \frac{d\nu}{d\mu} + \frac{d\mu}{d\mu} = \frac{d\nu}{d\mu} + 1$, we get that

$$\frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda} \cdot \frac{d\lambda}{d\mu} = \frac{d\nu}{d\lambda} \cdot \frac{d\nu}{d\mu} + \frac{d\nu}{d\lambda}.$$

So,

$$\frac{d\nu}{d\mu} \cdot \left(1 - \frac{d\nu}{d\lambda}\right) = \frac{d\nu}{d\lambda}.$$

By (c) we get that μ-a.e. $0 < 1 - \frac{d\nu}{d\lambda} \leq 1$, so we can divide by this function to complete the proof.

:) ✓

► **Exercise 13.27.** Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. Let $\nu = \mu\big|_{\mathcal{G}}$.

(a) Suppose that $f \in L^1(X, \mathcal{F}, \mu)$. Show that there exists $g \in L^1(X, \mathcal{G}, \nu)$ such that for every $A \in \mathcal{G}$,

$$\int_A f d\mu = \int_A g d\nu.$$

(b) Suppose that for $f \in L^1(X, \mathcal{F}, \mu)$ there are two such functions $g, g' \in L^1(X, \mathcal{G}, \nu)$ such that for all $A \in \mathcal{G}$,

$$\int_A f d\mu = \int_A g d\nu = \int_A g' d\nu.$$

Show that $g = g'$ ν-a.e.

♣ **Solution to ex:13.27.** :(

(a) Because $f \in L^1$, we know that $|f| < \infty$ μ-a.e., so we may assume that $|f| < \infty$.

First assume that f is positive and μ is finite. In this case, consider the function

$$\rho(A) := \int_A f d\mu$$

defined for all $A \in \mathcal{G}$. First of all, we showed in class that this defines a finite positive measure on $(X, \mathcal{G})$. Moreover, if $\nu(A) = 0$ for some $A \in \mathcal{G}$, since $\nu = \mu\big|_{\mathcal{G}}$ we have that $\mu(A) = 0$, and so $\rho(A) = \int_A f d\mu = 0$. Since this holds for all $A \in \mathcal{G}$, the signed measure ρ is absolutely continuous with respect to the measure ν. Since μ is finite, so is ν. Thus, by the Radon-Nykodim Theorem there exists a positive integrable $g = \frac{d\rho}{d\nu} \in L^1(X, \mathcal{G}, \nu)$ such that $d\rho = g d\nu$; that is, for all $A \in \mathcal{G}$,

$$\int_A f d\mu = \rho(A) = \int_A d\rho = \int_A g d\nu.$$

Now, if μ is only σ-finite, then write $X = \biguplus_n X_n$ with $\mu(X_n) < \infty$. Consider $\nu_n(A) := \mu(A \cap X_n)$ for all $A \in \mathcal{G}$. So $\nu = \sum_n \nu_n$. Define $\rho_n(A) := \int_{A \cap X_n} f d\mu$. Since

$$\sum_{j=1}^{n} f \mathbf{1}_{A \cap X_j} \nearrow f \mathbf{1}_A,$$

by monotone convergence we get that

$$\rho(A) := \int_A f d\mu = \sum_n \int_{A \cap X_n} f d\mu = \sum_n \rho_n(A).$$

Also, as above, if $\nu_n(A) = 0$ then $\mu(A \cap X_n) = 0$ and so $\rho_n(A) = 0$. So $\rho_n \ll \nu_n$. Since ν_n is finite, $g_n := \frac{d\rho_n}{\nu_n}$ exists and is in $L^1(X, \mathcal{G}, \nu_n)$. Specifically, g_n is $\mathcal{G}$-measurable. Also, since $\rho_n(A) = 0$ for $A \cap X_n = \emptyset$, we have that g_n can be chosen such that it is supported on X_n. Define $g = \sum_n g_n$. Since $(X_n)_n$ are disjoint and so g_n have disjoint support, we get that g is always finite and well defined. Also, since $g_n = g_n \mathbf{1}_{X_n}$, by monotone convergence again

$$\int_A g d\nu = \int \sum_n g_n \mathbf{1}_{A \cap X_n} d\nu = \sum_n \int_{A \cap X_n} g_n d\nu_n = \sum_n \rho_n(A) = \rho(A).$$

Specifically,

$$\int_X g d\nu = \rho(X) = \int_X f d\mu < \infty,$$

so $g \in L^1(X, \mathcal{G}, \nu)$.

Now, for the case that f is a general (not necessarily positive) function in L^1. Write $f = (f_1 - f_2) + i(f_3 - f_4)$ for $f_j \in L^1$ positive. By the previous case, there exist real-valued functions $g_j \in L^1(X, \mathcal{G}, \nu)$ such that for any $A \in \mathcal{G}$ and $j = 1, 2, 3, 4$ we have

$$\int_A f_j d\mu = \int_A g_j d\nu.$$

By linearity of the integral we get that for all $A \in \mathcal{G}$,

$$\int_A f d\mu = \int_A f_1 d\mu - \int_A f_2 d\mu + i \cdot \int_A f_3 d\mu - i \cdot \int_A f_4 d\mu = \int_A (g_1 - g_2) + i(g_3 - g_4) d\nu.$$

So we may choose $g = g_1 - g_2 + i(g_3 - g_4)$ which is a function in $L^1(X, \mathcal{G}, \nu)$.

(b) Suppose that g, g' are as in the question. Then for all $A \in \mathcal{G}$,

$$\int_A g d\nu = \int_A g' d\nu.$$

We have shown in class that this implies that $g = g'$ ν-a.e.

:) ✓

Number of exercises in lecture: 27

Total number of exercises until here: 156

Measure Theory

Ariel Yadin

Lecture 14: Convergence

Until now we have only considered the convergence of a sequence of functions $(f_n)_n$ to a limit f is a pointwise sense: $f_{\to} f$ means that $f_n(x) \to f$ for all x; we also extended this a bit to include a.e. convergence. We had a glimpse at some points of uniform convergence. The introduction of measure gives us many other topologies to consider.

14.1. Modes of convergence

• **Definition 14.1.** Let $(f_n)_n, f$ be a sequence of measurable functions on a measure space $(X, \mathcal{F}, \mu)$.

- We say that $(f_n)_n$ converges to f a.e., denoted $f_n \xrightarrow{\text{a.e.}} f$, if $\mu\{f_n \not\to f\} = 0$.
- We say that $(f_n)_n$ converges to f in L^1, denoted $f_n \xrightarrow{L^1} f$, if $\int |f_n - f| d\mu \to 0$ as $n \to \infty$.
- We say that $(f_n)_n$ is **Cauchy in measure** if for every $\varepsilon > 0$ $\mu\{|f_n - f_m| > \varepsilon\} \to 0$ as $m, n \to \infty$.
- We say that $(f_n)_n$ converges to f in measure, denoted $f_n \xrightarrow{\mu} f$, if for every $\varepsilon > 0$ $\mu\{|f_n - f| > \varepsilon\} \to 0$ as $n \to \infty$.

Example 14.2. Let us consider a few examples on $(\mathbb{R}, \mathcal{B}, \lambda)$. $f_n = \frac{1}{n}\mathbf{1}_{[0,n]}$, $g_n = \mathbf{1}_{[n,n+1]}$, $\ell_n = n\mathbf{1}_{[0,1/n]}$ and

$$h_n = \mathbf{1}_{[j2^{-k},(j+1)2^{-k}]} \qquad \text{for } n = 2^k + j \ , \ 0 \le j < 2^k.$$

(That is, $k = \lfloor \log_2 n \rfloor$ and $j = n - 2^k$.) So that

$$h_1 = \mathbf{1}_{[0,1]} \qquad h_2 = \mathbf{1}_{[0,\frac{1}{2}]} \qquad h_3 = \mathbf{1}_{[\frac{1}{2},1]} \qquad \text{etc.}$$

We have

$$f_n \xrightarrow{\text{a.e.}} 0 \qquad g_n \xrightarrow{\text{a.e.}} 0 \qquad \ell_n \xrightarrow{\text{a.e.}} 0.$$

However

$$\int |f_n| d\lambda = \int |g_n| d\lambda = \int |\ell_n| d\lambda = 1.$$

So $f_n \stackrel{L^1}{\not\longrightarrow} 0, g_n \stackrel{L^1}{\not\longrightarrow} 0, \ell_n \stackrel{L^1}{\not\longrightarrow} 0$.

As for h_n, note that $\int |h_n| d\lambda = 2^{-\lfloor \log_2 n \rfloor}$. So $h_n \xrightarrow{L^1} 0$. However, $(h_n)_n$ does not converge pointwise, because for any $x \in (0,1)$ there are infinitely many n such that $h_n(x) = 0$ and infinitely many n for which $h_n(x) = 1$.

Note that $(g_n)_n$ is not Cauchy in measure. Indeed, $|g_n(x) - g_m(x)| > \varepsilon$ if and only if $x \in [n, n+1] \triangle [m, m+1]$. So

$$\lambda \{|g_n - g_m| > \varepsilon\} = \lambda([n, n+1] \triangle [m, m+1]) = 2.$$

It is also a sequence that does not converge in measure, as we will see later.

On the other hand, $f_n \xrightarrow{\lambda} 0, \ell_n \xrightarrow{\lambda} 0$ and $h_n \xrightarrow{\lambda} 0$: For all $n > \frac{1}{\varepsilon}$ we have that $|f_n| \leq \frac{1}{n} < \varepsilon$, so $\lambda \{|f_n| > \varepsilon\} = \lambda(\emptyset) = 0$. $\ell_n(x) > \varepsilon$ if and only if $x \in [0, 1/n]$ and $n > \varepsilon$. So

$$\lim_{n \to \infty} \lambda \{|\ell_n| > \varepsilon\} = \lim_{n \to \infty} \lambda([0, 1/n]) = 0.$$

$|h_n(x)| > \varepsilon$ if and only if $x \in [j2^{-k}, (j+1)2^{-k}]$ for $k = \lfloor \log_2 n \rfloor$ and $j = n - 2^k$. So

$$\lambda \{|h_n| > \varepsilon\} = \lambda([j2^{-k}, (j+1)2^{-k}]) = 2^{-\lfloor \log_2 n \rfloor} \to 0.$$

To sum up:

	a.e.	L^1	measure
f_n	✓	X	✓
g_n	✓	X	X
ℓ_n	✓	X	✓
h_n	X	✓	✓

△▽△

By these examples, it is not always true that a.e. convergence implies L^1 convergence, or vice-versa. Recall however the Dominated Convergence Theorem which actually relates a.e. convergence to L^1 convergence.

► **Exercise 14.1.** Show that if $f_n \xrightarrow{\text{a.e.}} f$ and $|f_n| \leq g \in L^1$ for all n then $f \in L^1$ and $f_n \xrightarrow{L^1} f$.

Show that if $f_n \xrightarrow{\text{a.e.}} f$ are bounded functions and the measure space is finite, then $f_n \xrightarrow{L^1} f$.

• **Proposition 14.3** (L^1 implies measure)**.** *If $f_n \xrightarrow{L^1} f$ then $f_n \to \mu\mu f$.*

Proof. Set $A_{n,\varepsilon} = \{|f_n - f| > \varepsilon\}$. Then,

$$\varepsilon \cdot \mu(A_{n,\varepsilon}) \leq \int_{A_{n,\varepsilon}} |f_n - f| d\mu \leq \int |f_n - f| d\mu \to 0.$$

□

The converse is false as is seen by f_n, ℓ_n above.

• **Proposition 14.4** (a.e. implies measure)**.** *If $f_n \xrightarrow{a.e.} f$ then $f_n \to \mu\mu f$.*

Proof. For any $\varepsilon > 0$ let $A_{n,\varepsilon} = \{|f_n - f| > \varepsilon\}$. Note that if $x \in A_{n,\varepsilon}$ then $f_n(x) \not\to f(x)$. So $A_{n,\varepsilon} \subset N$ where $N = \{x \ : \ f_n(x) \not\to f(x)\}$. Thus, $\mu(A_{n,\varepsilon}) \leq \mu(N) = 0$. □

The converse is false as shown by h_n above.

However, we can still relate convergence in measure to a.e. convergence of a subsequence.

• **Proposition 14.5.** *$(f_n)_n$ is Cauchy in measure if and only if there is a measurable function f such that $f_n \xrightarrow{\mu} f$.*

Moreover, if $f_n \xrightarrow{\mu} f$ then there exists a subsequence $(f_{n_k})_k$ such that $f_{n_k} \xrightarrow{a.e.} f$ a.e. as $k \to \infty$.

Proof. Suppose that $(f_n)_n$ is Cauchy in measure. For any k there exists n_k such that for all $n, m \geq n_k$, $\mu\left\{|f_n - f_m| > 2^{-k}\right\} < 2^{-k}$. Let $g_k = f_{n_k}$. Set

$$F = \limsup_k \left\{|f_{n_k} - f_{n_k+1}| > 2^{-k}\right\} = \bigcap_n \bigcup_{k \geq n} \left\{|f_{n_k} - f_{n_k+1}| > 2^{-k}\right\}.$$

Since for any n,

$$\mu(F) \leq \mu\Big(\bigcup_{k \geq n} \left\{|f_{n_k} - f_{n_k+1}| > 2^{-k}\right\} \Big) \leq \sum_{k \geq n} 2^{-k} = 2^{-n+1},$$

we have that $\mu(F) = 0$. Note that for any $x \notin F$ we have that $(g_k(x))_k$ is a Cauchy sequence in $\mathbb{C}$. So we may define $f(x) = \lim_{k\to\infty} g_k(x)$ for $x \notin F$ and $f(x) = 0$ for $x \in F$. We had an exercise proving that such a function is measurable. Since $\mu(F) = 0$ we have that $f_{n_k} = g_k \xrightarrow{\text{a.e.}} f$, and so $g_k \xrightarrow{\mu} f$. But then,

$$\mu\{|f_n - f| > \varepsilon\} \le \mu\big(\{|f_n - g_k| > \varepsilon/2\} \bigcup \{|g_k - f| > \varepsilon/2\}\big)$$
$$\le \mu\{|f_n - g_k| > \varepsilon/2\} + \mu\{|g_k - f| > \varepsilon/2\} \to 0 \qquad \text{as } n,k \to \infty.$$

So $f_n \xrightarrow{\mu} f$.

For the other direction, if $f_n \xrightarrow{\mu} f$ then for any $\varepsilon > 0$,

$$\mu\{|f_n - f_m| > \varepsilon\} \le \mu\{|f_n - f| > \varepsilon/2\} + \mu\{|f_m - f| > \varepsilon/2\} \to 0 \qquad \text{as } m,n \to \infty.$$

So $(f_n)_n$ is Cauchy in measure. □

► **Exercise 14.2.** Show that if $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\mu} g$ then $f = g$ a.e.

Conclude that if $(f_n)_n$ converges to f and to g in any one of the three modes (a.e., L^1, measure), possibly different modes for f and for g, then $f = g$ a.e.

♣ **Solution to ex:14.2.** :(

If $(f_n)_n$ converges to f and g then $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\mu} g$. So it suffices to work with this assumption.

For any $m > 0$,

$$\mu\left\{|f - g| > \tfrac{1}{m}\right\} \le \mu\left\{|f - f_n| > \tfrac{1}{m}\right\} + \mu\left\{|g - f_n| > \tfrac{1}{m}\right\} \to 0 \qquad \text{as } n \to \infty.$$

So

$$\mu\{f \neq g\} \le \sum_m \mu\left\{|f - g| > \tfrac{1}{m}\right\} = 0.$$

:) ✓

► **Exercise 14.3.** Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $(f_n)_n$ be a sequence of non-negative measurable functions, and let f be a measurable function such that $(f_n)_n$ converges to f in **measure**.

Show that

$$\int f d\mu \leq \liminf_{n\to\infty} \int f_n d\mu.$$

♣ **Solution to ex:14.3.** :(

Let $(f_{n_k})_k$ be a subsequence such that

$$\lim_{k\to\infty} \int f_{n_k} d\mu = \liminf_n \int f_n d\mu.$$

So we want to show that $\int f d\mu \leq \lim_{k\to\infty} \int f_{n_k} d\mu$. Since $(f_{n_k})_k$ is a subsequence, we have that for all $\varepsilon > 0$,

$$\lim_{k\to\infty} \mu\{|f_{n_k} - f| > \varepsilon\} \leq \limsup_n \mu\{|f_n - f| > \varepsilon\} = 0.$$

So $(f_{n_k})_k$ converges in measure to f.

Let $g_k := f_{n_k}$ for all k, which convergen in measure to f. By a theorem in class we now have that there is a further subsequence $(g_{k_j})_j$ such that $\lim_{j\to\infty} g_{k_j} = f$ a.e. Since these are all non-negative functions, Fatou's Lemma tells us that

$$\int f d\mu \leq \liminf_j \int g_{k_j} d\mu.$$

However, the sequence $(\int g_{k_j} d\mu)_j$ is a subsequence of the converging sequence $(\int f_{n_k} d\mu)_k$ which converges to $\liminf_n \int f_n d\mu$. So the limit is

$$\int f d\mu \leq \liminf_j \int g_{k_j} d\mu \leq \lim_{j\to\infty} \int g_{k_j} d\mu = \lim_{k\to\infty} \int f_{n_k} d\mu = \liminf_n \int f_n d\mu.$$

:) ✓

► **Exercise 14.4.** Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $(f_n)_n$ be a sequence of measurable functions, and let f be a measurable function such that $(f_n)_n$ converges to f in **measure**.

Suppose that $g \in L^1(X, \mathcal{F}, \mu)$ such that for all n, $|f_n| \leq g$.

Show that $(f_n)_n$ converges to f in L^1.

♣ **Solution to ex:14.4.** :(

Since $(f_n)_n$ converges to f in measure, there is a subsequence $(n_k)_k$ such that $(f_{n_k})_k$ converges to f μ-a.e. Since $|f_n| \leq g$ for all n, we get that $|f_{n_k} - f| \to 0$ μ-a.e. as $k \to \infty$. Also $|f| = \lim_k |f_{n_k}| \leq g$. Thus, $|f_n - f| \leq 2g$ for all n.

Consider the sequence $I_n := \int |f_n - f| d\mu$. We want to show that $\limsup_{n\to\infty} I_n = 0$.

Let $(f_{n_k})_k$ be a subsequence such that $I_{n_k} \to \limsup_n I_n$ as $k \to \infty$. Set $h_k := |f_{n_k} - f|$. Since this is a subsequence, we have that for all $\varepsilon > 0$,

$$\lim_{k\to\infty} \mu \{h_k > \varepsilon\} \leq \limsup_{n\to\infty} \mu \{|f_n - f| > \varepsilon\} = 0.$$

So $(h_k)_k$ converges to 0 in measure.

We saw in class that there exists a subsequence $(h_{k_j})_j$ such that $\lim_{j\to\infty} h_{k_j} = 0$ μ-a.e.

Because $(h_{k_j})_j$ is a subsequence of $(h_k)_k = (|f_{n_k} - f|)_k$ which is a subsequence of $(|f_n - f|)_n$, we have that $h_{k_j} \leq 2g \in L^1$ for all j. Since $(h_{k_j})_j$ converges to 0 a.e. and is a dominated sequence, we have by the dominated convergence theorem that

$$\lim_{j\to\infty} \int h_{k_j} d\mu = 0.$$

But by definition, $(h_{k_j})_j$ is a subsequence of $(|f_{n_k} - f|)_k$; so the sequence $(\int h_{k_j} d\mu)_j$ is a subsequence of the converging sequence $(I_{n_k})_k$. Thus these sequences have the same limit which is

$$\limsup_n I_n = \lim_{k\to\infty} I_{n_k} = \lim_{j\to\infty} \int h_{k_j} d\mu = 0.$$

:) ✓

► **Exercise 14.5.** [Folland p.63] Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $(A_n)_n$ be a sequence of measurable sets with finite measure. Suppose that $\mathbf{1}_{A_n} \xrightarrow{L^1} f$.

Show that $f = \mathbf{1}_A$ a.e. for some $A \in \mathcal{F}$.

• **Definition 14.6.** Let $(f_n)_n, f$ be a sequence of measurable functions on a measure space $(X, \mathcal{F}, \mu)$.

We say that $(f_n)_n$ converges uniformly to f if $\sup_x |f_n(x) - f(x)| \to 0$ as $n \to \infty$; that is, for every $\varepsilon > 0$ there exists n_0 such that for all $n > n_0$ and any x, $|f_n(x) - f(x)| < \varepsilon$.

We say that $(f_n)_n$ converges uniformly to f on A if $(f_n \mathbf{1}_A)_n$ converges uniformly to $f\mathbf{1}_A$; that is $\sup_{x \in A} |f_n(x) - f(x)| \to 0$ as $n \to \infty$.

We say that $(f_n)_n$ converges **almost uniformly** to f if for every $\varepsilon > 0$ there exists a measurable A such that $\mu(A^c) < \varepsilon$ and $(f_n)_n$ converges uniformly to f on A.

••• **Theorem 14.7** (Egoroff's Theorem)**.** *Let $(X, \mathcal{F}, \mu)$ be a* finite *measure space. Suppose that $f_n \xrightarrow{a.e.} f$. Then $(f_n)_n$ converges almost uniformly to f.*

Proof. By augmenting f_n, f on a set of measure 0 we may assume without loss of generality that $f_n(x) \to f(x)$ for every x.

Let
$$B_{n,k} = \bigcup_{m \geq n} \left\{ |f_m - f| > \tfrac{1}{k} \right\}.$$
For any $x \in B_k := \bigcap_n B_{n,k}$ we have that for all n there exists $m \geq n$ such that $|f_m(x) - f(x)| > \frac{1}{k}$. That is, $\limsup_{n \to \infty} |f_n(x) - f(x)| \geq \frac{1}{k} > 0$. So $f_n(x) \not\to f(x)$. That is, $B_k \subset \{f_n \not\to f\}$.

For fixed k, the sequence $(B_{n,k})_n$ is decreasing. Since μ is a finite measure,
$$\lim_{n \to \infty} \mu(B_{n,k}) = \mu(B_k) \leq \mu \{f_n \not\to f\} = 0.$$

For any $\varepsilon > 0$ and any $k \geq 1$ let $n_{k,\varepsilon}$ be large enough so that $\mu(B_{n_{k,\varepsilon},k}) < \varepsilon 2^{-k}$. Let $B = \bigcup_k B_{n_{k,\varepsilon},k}$. Then $\mu(B) \leq \sum_k \mu(B_{n_{k,\varepsilon},k}) \leq \varepsilon$. For $A = B^c$ we have that for any $\eta > 0$ taking $k = \lceil \eta^{-1} \rceil$, there exists $n_0 = n_{k,\varepsilon}$ such that for all $n \geq n_0$ we have that for all $x \in A$, $|f_n(x) - f(x)| \leq \frac{1}{k} \leq \eta$. Thus, $(f_n)_n$ converges uniformly to f on A.

To conclude: for any $\varepsilon > 0$ we can find A such that $(f_n)_n$ converges uniformly to f on A and $\mu(A^c) < \varepsilon$. This is almost uniform convergence. □

▶ **Exercise 14.6.** Show that if $(f_n)_n$ converges almost uniformly to f then $f_n \xrightarrow{\text{a.e.}} f$.

▶ **Exercise 14.7.** A version of Egoroff's theorem in non-finite settings:

Let $(X, \mathcal{F}, \mu)$ be a measure space. Suppose that $f_n \xrightarrow{\text{a.e.}} f$. Suppose that $|f_n| \leq g \in L^1$. Show that $(f_n)_n$ converges almost uniformly to f.

▶ **Exercise 14.8.** Suppose that $(X, \mathcal{F}, \mu)$ is a σ-finite measure space. Suppose that $f_n \xrightarrow{\text{a.e.}} f$.

Show that there exists a sequence of measurable sets $(A_k)_k$, such that $\mu(\bigcap_k A_k^c) = 0$ and such that for any k, $(f_n)_n$ converges uniformly to f on A_k.

▶ **Exercise 14.9.** Show that if $(f_n)_n$ are continuous functions into $\mathbb{C}$ on some Borel measure space, and if $(f_n)_n$ converges to f uniformly, then f is continuous as well.

♣ **Solution to ex:14.9.** :(

Let $\varepsilon > 0$. Take $n_0 = n_0(\varepsilon)$ so that for all $n > n_0$ we have $\sup_x |f_n(x) - f(x)| < \varepsilon/2$.

Now, if $(x_k)_k$ is any sequence $x_k \to x$, then $f_n(x_k) \to f_n(x)$ for all n. For all $n > n_0(\varepsilon)$,

$$|f(x_k) - f(x)| \leq |f(x_k) - f_n(x_k)| + |f(x) - f_n(x)| + |f_n(x_k) - f_n(x)|$$
$$\leq 2 \sup_x |f(x) - f_n(x)| + |f_n(x_k) - f_n(x)| < \varepsilon + |f_n(x_k) - f_n(x)|.$$

Taking $k \to \infty$ we have that for any $\varepsilon > 0$ $\limsup_k |f(x_k) - f(x)| \leq \varepsilon$. Thus, f is continuous at x, for all x. :) ✓

• **Lemma 14.8.** *Let $A \subset \mathbb{R}$ be a Lebesgue set of finite Lebesgue measure. For any $\varepsilon > 0$ there exists a continuous function ψ such that $0 \leq \psi \leq 1$, ψ vanishes outside an interval and*

$$\int |\psi - \mathbf{1}_A| d\lambda < \varepsilon.$$

Proof. For any interval $I = [a, b]$, let

$$\psi_{\varepsilon,I}(x) = \begin{cases} 0 & x \notin (a,b), \\ 1 & x \in [a+\varepsilon, b-\varepsilon], \\ \frac{x-a}{\varepsilon} & x \in [a, a+\varepsilon], \\ \frac{b-x}{\varepsilon} & x \in [b-\varepsilon, b]. \end{cases}$$

So $\psi_{\varepsilon,I}$ is continuous and vanishes outside I, and is 1 in $[a+\varepsilon, b-\varepsilon]$ and $0 \leq \psi(x) \leq 1$ for $x \in [a,b] \setminus [a+\varepsilon, b-\varepsilon]$. Thus, $\int |\mathbf{1}_I - \psi_{\varepsilon,I}| d\lambda \leq 2\varepsilon$.

If $A \subset [a,b]$ then $\lambda(A) = \sup_{A \supset K \text{ compact}} \lambda(K)$. Fix $\varepsilon > 0$. Choose a compact $K \subset A$ so that $\lambda(K) \geq \lambda(A) - \varepsilon$. Now we may find a sequence of almost disjoint closed intervals (*i.e.* with disjoint interiors) $(I_n)_n$ such that $\lambda(I_n) \leq \lambda(K) + \varepsilon$ and $K \subset \bigcup_n I_n$. We may assume without loss of generality that $I_n \cap K \neq \emptyset$ for all n.

For every n let I'_n be the open $\varepsilon \cdot 2^{-n}$ enlargement of I_n; that is, if $I_n = [x_n, y_n]$ then $I'_n = (x_n - \varepsilon \cdot 2^{-n}, y_n + \varepsilon \cdot 2^{-n})$. So $K \subset \bigcup_n I'_n$ is an open cover. So there exists n such that $K \subset \bigcup_{j=1}^n I'_j$. Note that

$$\lambda(\bigcup_{j=1}^n I'_j) \leq \sum_{j=1}^n \lambda(I'_j) \leq \varepsilon + \sum_{j=1}^\infty \lambda(I_n) \leq \lambda(K) + 2\varepsilon.$$

For every $j = 1, \ldots, n$ let $\psi_j = \psi_{\varepsilon \cdot 2^{-j}, I_j}$. So $\int |\psi_j - \mathbf{1}_{I_j}| d\lambda \leq 2 \cdot 2^{-j}$. Thus for $\varphi = \sum_{j=1}^n \mathbf{1}_{I_j}$ and $\psi = \sum_{j=1}^n \psi_j$ we have that

$$\int |\psi - \varphi| d\lambda \leq \sum_{j=1}^n \int |\psi_j - \mathbf{1}_{I_j}| d\lambda \leq \sum_{j=1}^n \varepsilon 2^{-j} \leq \varepsilon.$$

Also,

$$\begin{aligned}\int |\mathbf{1}_A - \varphi| d\lambda &\le \sum_{j=1}^{n} \int |\mathbf{1}_{K\cap I_n} - \mathbf{1}_{I_n}| d\lambda + \int |\mathbf{1}_A - \mathbf{1}_K| d\lambda \\ &\le \sum_{j=1}^{n} \lambda(I_n \setminus K) + \lambda(A \setminus K) \le \lambda(\bigcup_{j=1}^{n} I_j \setminus K) + \lambda(A \setminus K) \\ &\le \lambda(\bigcup_{j=1}^{n} I'_j) - \lambda(K) + \lambda(A) - \lambda(K) \le 3\varepsilon.\end{aligned}$$

Note that ψ is a function that vanishes outside $\bigcup_{j=1}^{n} I_j$, which is contained in some interval $I = [a, b]$. □

► **Exercise 14.10.** Show that if $f : \mathbb{R} \to \mathbb{C}$ is Lebesgue and integrable then for every $\varepsilon > 0$ there exists a continuous function ψ such that ψ vanishes outside a bounded interval and

$$\int |f - \psi| d\lambda < \varepsilon.$$

♣ **Solution to ex:14.10.** :(

If $f = \sum_{j=1}^{n} a_j \mathbf{1}_{A_j}$ is a simple function, where $(A_j)_{j=1}^{n}$ are pairwise disjoint, the for each j find a continuous function ψ_j vanishing outside some bounded interval such that $\int |\psi_j - \mathbf{1}_{A_j}| d\lambda < \frac{\varepsilon}{n a_j}$. Then, for $\psi = \sum_{j=1}^{n} a_j \psi_j$ we have that

$$\int |f - \psi| d\lambda \le \sum_{j=1}^{n} a_j \int |\mathbf{1}_{A_j} - \psi_j| d\lambda < \varepsilon.$$

Note that ψ vanishes outside some interval (which just is the smallest interval containing the union of the bounded intervals supporting the ψ_j's).

Now if f is any non-negative measurable function, let $\varphi_n \nearrow f$ be an approximating sequence of simple functions. By Dominated Convergence, $\int |\varphi_n - f| d\lambda \to 0$. So there exist n large enough so that $\int |\varphi_n - f| d\lambda < \frac{\varepsilon}{2}$. for this n let ψ_n be a continuous function supported in a bounded interval such that $\int |\varphi_n - \psi_n| d\lambda < \frac{\varepsilon}{2}$. Note that

$$\int |\psi_n - f| d\lambda \le \int |\varphi_n - f| d\lambda + \int |\psi_n - \varphi_n| d\lambda < \varepsilon.$$

Now if f is a general Lebesgue measurable function (into $\mathbb{C}$), write $f = f_1 - f_2 + i(f_3 - f_4)$, for non-negative Lebesgue f_j. Then choose continuous functions ψ_j that are supported in a bounded interval and $\int |f_j - \psi_j| d\lambda < \varepsilon/4$. Summing we have that $\psi = \psi_1 - \psi_2 + i(\psi_3 - \psi_4)$ is supported in a bounded interval and continuous, and

$$\begin{aligned}\int |f - \psi| d\lambda &\le \int \sqrt{|f_1 - f_2 - (\psi_1 - \psi_2)|^2 + |f_3 - f_4 - (\psi_3 - \psi_4)|^2} d\lambda \\ &\le \int |f_1 - f_2 - (\psi_1 - \psi_2)| d\lambda + \int |f_3 - f_4 - (\psi_3 - \psi_4)| d\lambda \\ &\le \sum_{j=1}^{4} \int |f_j - \psi_j| d\lambda < \varepsilon.\end{aligned}$$

:) ✓

●●● **Theorem 14.9** (Lusin's Theorem)**.** *Let $f : [a, b] \to \mathbb{C}$ be Lebesgue. Let $\varepsilon > 0$. Then there exists a compact set $K \subset [a, b]$ such that $\lambda([a, b] \setminus K) < \varepsilon$ and $f\big|_K$ is continuous.*

Proof. Let $(\psi_n)_n$ be a sequence of continuous functions each supported in a bounded closed interval $I_n \subset [a, b]$ such that $\int |f - \psi_n| d\lambda < \frac{1}{n}$. Thus, $\psi_n \xrightarrow{L^1} f$, and so $\psi_n \xrightarrow{\lambda} f$. Let $(g_k = \psi_{n_k})_k$ be a subsequence such that $g_k \xrightarrow{\text{a.e.}} f$. Since $\lambda([a, b]) < \infty$,

Egoroff's Theorem tells us that $(g_k)_k$ converges almost uniformly to f. That is, for any $\varepsilon > 0$ there exists a Lebesgue set A_ε such that $\lambda([a, b] \setminus A_\varepsilon) < \varepsilon/2$ and $(g_k)_k$ converges uniformly to f on A_ε. Let $K_\varepsilon \subset A_\varepsilon$ be a compact subset such that $\lambda(A_\varepsilon \setminus K_\varepsilon) < \varepsilon/2$. So $\lambda([a, b] \setminus K_\varepsilon) \le \lambda([a, b] \setminus A_\varepsilon) + \lambda(A_\varepsilon \setminus K_\varepsilon) < \varepsilon$. Also, $(g_k)_k$ converges uniformly to f on K_ε, so f is continuous on K_ε. □

Lusin's Theorem is informally the statement that Lebesgue functions are almost continuous.

Number of exercises in lecture: 10

Total number of exercises until here: 166

Measure Theory

Ariel Yadin

Lecture 15: Differentiation

We now turn to understanding differentiation in $\mathbb{R}^d$. We will work throughout in the measure space $(\mathbb{R}^d, \mathcal{B}_d, \lambda = \lambda^d)$. $B(x,r)$ denotes the open ball of radius r around x (in L^2-norm).

15.1. Hardy-Littlewood Maximal Theorem

A technical lemma:

• **Lemma 15.1.** *Let $U = \bigcup_\alpha B_\alpha$ be a union of a family of open balls $B_\alpha = B(x_\alpha, r_\alpha)$ for every α. Then, for any $c < \lambda(U)$ there exist finitely many pairwise disjoint balls $B_j = B_{\alpha_j}$, $j = 1, \ldots, n$, such that $\sum_{j=1}^n \lambda(B_j) > 3^{-d}c$.*

Proof. There exists a compact $K \subset U$ such that $\lambda(K) > c$, by inner regularity. Since $K \subset \bigcup_\alpha B_\alpha$ is an open cover, there exists finitely many $A_j = B_{\alpha_j}$, $j = 1, \ldots, m$, that cover the compact K, $K \subset \bigcup_{j=1}^m A_j$.

Reorder $A_1, \ldots, A_m$ so that the radii are decreasing. Set $B_1 = A_1$ and for all $j \geq 2$ let B_j be A_k for the smallest k such that A_k is disjoint from $B_1 \cup \cdot \cup B_{j-1}$.

Suppose that $B_j = B(x_j, r_j)$ for all j. These are disjoint by definition.

Now, if $x \in A_k$, the for some j we have that $A_k \cap B_j \neq \emptyset$. If j is the smallest such possible index, then A_k is disjoint from $B_1 \cup \cdot \cup B_{j-1}$, so the radius of A_k is at most that of B_j; otherwise we would have wanted to choose A_k instead of B_j. Thus, since there is some $x \in A_k \cap B_j$, we have that all elements in A_j are at distance at most $2\text{rad}(A_k) \leq 2r_j$ from x, and so $A_j \subset B(x, 2r_j) \subset B(x_j, 3r_j)$.

Since $K \subset \bigcup_{j=1}^m A_j \subset \bigcup_{j=1}^n B(x_j, 3r_j)$ we have that

$$c < \lambda(K) \leq \sum_{j=1}^n \lambda(B(x_j, 3r_j)) = 3^d \sum_{j=1}^n \lambda(B_j).$$

$\square$

• **Definition 15.2.** A measurable function $f:\mathbb{R}^d\to\mathbb{C}$ is called **locally integrable** if for every bounded measurable set $B\subset\mathbb{R}^d$, $\int_B |f|d\lambda<\infty$.

$L^1_{\text{loc}}=L^1_{\text{loc}}(\mathbb{R}^d,\mathcal{B}_d,\lambda)$ denotes the space of locally integrable functions.

For any $x\in\mathbb{R}^d$ and $r>0$ we define the average of $f\in L^1_{\text{loc}}$ on $B(x,r)$ by

$$(A_rf)(x)=\frac{1}{\lambda(B(x,r))}\int_{B(x,r)} fd\lambda.$$

• **Proposition 15.3.** *For any $f\in L^1_{\text{loc}}$, A_rf is jointly continuous in x,r.*

Proof. Let $x_n\to x, r_n\to r$.

Then $\mathbf{1}_{B(x_n,r_n)}(x)\to\mathbf{1}_{B(x,r)}$ for all $x\notin B[x,r]\setminus B(x,r)$. Since $\lambda(B[x,r]\setminus B(x,r))=0$ we get that $f\mathbf{1}_{B(x_n,r_n)}\xrightarrow{\text{a.e.}} f\mathbf{1}_{B(x,r)}$. Also, since $f\in L^1_{\text{loc}}$, for all n such that $|r_n-r|<1, \text{dist}(x_n,x)<1$ we have that $|f\mathbf{1}_{B(x_n,r_n)}|\le|f\mathbf{1}_{B(x,r+2)}|\in L^1$. So by Dominated Convergence,

$$\int_{B(x_n,r_n)} fd\lambda\to\int_{B(x,r)} fd\lambda \qquad\text{and}\qquad \lambda(B(x_n,r_n))\to\lambda(B(x,r)).$$

Thus, $A_{r_n}f(x_n)\to A_rf(x)$. □

This implies that A_rf is a measurable function.

► **Exercise 15.1.** Let $x_n\to x, r_n\to r$. Show that $\mathbf{1}_{B(x_n,r_n)}(x)\to\mathbf{1}_{B(x,r)}$ for all $x\notin B[x,r]\setminus B(x,r)$.

► **Exercise 15.2.** Show that $\lambda(B[x,r])=\lambda(B(x,r))$.

• **Definition 15.4.** Given $f\in L^1_{\text{loc}}$ define the **Hardy-Littlewood maximal function**

$$Mf(x)=\sup_{r>0}A_r|f|(x)=\sup_{r>0}\frac{1}{\lambda(B(x,r))}\int_{B(x,r)}|f|d\lambda.$$

▶ **Exercise 15.3.** Show that Mf is measurable.

●●● **Theorem 15.5** (Hardy-Littlewood Maximal Theorem)**.** *For all* $\alpha > 0$ *and all* $f \in L^1$,

$$\lambda(Hf > \alpha) = \lambda(\{x \ : \ Hf(x) > \alpha\}) \leq \frac{3^d}{\alpha} \int |f| d\lambda.$$

Proof. Let $U = \{Hf > \alpha\}$. For every $x \in U$ there exists $r_x > 0$ such that $\int_{B(x,r_x)} |f| d\lambda > \alpha \cdot \lambda(B(x, r_x))$. Since $U \subset_{x \in U} B(x, r_x)$, for any $c < \lambda(U)$ there exist $x_1, \ldots, x_n$ such that for $r_j = r_{x_j}$ the balls $(B_j := B(x_j, r_j))_{j=1}^n$ are pairwise disjoint and $\sum_{j=1}^n \lambda(B_j) > 3^{-d} c$. Thus,

$$c < 3^d \sum_{j=1}^n \lambda(B_j) \leq 3^d \frac{1}{\alpha} \sum_{j=1}^n \int_{B(x_j, r_j)} |f| d\lambda \leq \frac{3^d}{\alpha} \cdot \int |f| d\lambda.$$

Taking supremum of $c < \lambda(U)$ we obtain the result. □

▶ **Exercise 15.4.** Recall for a function $\psi : \mathbb{R} \to \mathbb{R}$, the definitions

$$\limsup_{x \to y} \psi(x) := \lim_{\varepsilon \to 0} \sup_{0 < |x-y| < \varepsilon} \psi(x) = \inf_{\varepsilon > 0} \sup_{0 < |x-y| < \varepsilon} \psi(x).$$

Show that $\limsup_{x \to y} |\psi(x) - c|$ if and only if for every sequence $x_n \to y$ we have $\psi(x_n) \to c$. (In this case we say that $\lim_{x \to y} \psi(x) = c$.)

●●● **Theorem 15.6** (Basic Differentiation Theorem)**.** *For any* $f \in L^1_{\text{loc}}$,

$$\lim_{r \to 0} A_r f(x) = f(x) \qquad \text{for } \lambda\text{-a.e. } x.$$

Proof. First assume that $f = f \mathbf{1}_{B(0,R)}$ for some $R > 0$. So $f \in L^1$.

For every $\varepsilon > 0$ we can find a continuous function g such that $\int |f - g| d\lambda < \varepsilon$. So for any x and $\delta > 0$ there exists $r > 0$ such that if $|y - x| < r$ then $|g(x) - g(y)| < \delta$. For this $r > 0$,

$$|A_r g(x) - g(x)| \leq \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |g(y) - g(x)| d\lambda(y) \leq \delta.$$

Thus, $\lim_{r\to 0} A_r g(x) = g(x)$ for all x. We conclude that

$$\limsup_{r\to 0} |A_r f(x) - f(x)| \leq \limsup_{r\to 0} (|A_r(f-g)(x)| + |A_r g(x) - g(x)| + |g(x) - f(x)|$$
$$\leq H(f-g)(x) + |f-g|(x).$$

So setting

$$E_\alpha = \left\{ x \ : \ \limsup_{r\to 0} |A_r f(x) - f(x)| > \alpha \right\},$$

we have that $E_\alpha \subset \{Hf > \alpha/2\} \cup \{|f-g| > \alpha/2\}$. We have

$$\alpha\lambda(\{|f-g| > \alpha\}) \leq \int_{\{|f-g|>\alpha\}} |f-g| d\lambda \leq \int |f-g| d\lambda < \varepsilon,$$

so by the Hardy-Littlewood Maximal Theorem,

$$\lambda(E_\alpha) \leq \frac{3^d \varepsilon}{\alpha} + \frac{2\varepsilon}{\alpha} \to 0 \qquad \text{as } \varepsilon \to 0.$$

Set $E = \bigcup_n E_{1/n}$. Then $\lambda(E) = 0$ and for any $x \notin E$, $\lim_{r\to 0} A_r f(x) = f(x)$.

This proves the theorem for f that vanishes outside $B(0,R)$.

For general $f \in L^1_{\text{loc}}$, then for all $x \in B(x,R)$ and $r < R$ we have that $A_r f(x) = A_r g(x)$ where $g = f\mathbf{1}_{B(0,2R)}$. So for a.e. $x \in B(0,R)$,

$$\lim_{r\to 0} A_r f(x) = \lim_{R>r\to 0} A_r f(x) = \lim_{r\to 0} A_r g(x) = g(x) = f(x).$$

So for every R there is a set $N_R \subset B(0,R)$ such that $\lambda(N_R) = 0$ and for $x \in B(0,R) \setminus N_R$ we have $\lim_{r\to 0} A_r f(x) = f(x)$. Set $N = \bigcup_R N_R$. Then $\lambda(N) = 0$ and if $x \notin N$ then $\lim_{r\to\infty} A_r f(x) = f(x)$. □

15.2. The Lebesgue Differentiation Theorem

• **Definition 15.7.** For a function $f \in L^1_{\text{loc}}$ define the **Lebesgue set of** f by

$$L_f = \left\{ x \ : \ \tfrac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| d\lambda(y) = 0 \right\}.$$

• **Proposition 15.8.** *For any* $f \in L^1_{\text{loc}}$, $\lambda(L_f^c) = 0$.

Proof. For any $z \in \mathbb{Q} + i\mathbb{Q} \subset \mathbb{C}$, we have a null set N_z such that for $x \notin N_z$, by applying the Basic Differentiation Theorem to the function $y \mapsto |f(y) - z|$,

$$\lim_{r \to 0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y) - z| f \lambda(y) = 0.$$

Let $N = \bigcup_{z \in \mathbb{Q}+i\mathbb{Q}} N_z$. So $\lambda(N) = 0$.

Let $x \notin N$. Fix $\varepsilon > 0$. Let $z \in \mathbb{Q} + i\mathbb{Q}$ be such that $|f(x) - z| < \varepsilon$. So $|f(y) - f(x)| \leq |f(y) - z| + \varepsilon$, and

$$\frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| d\lambda(y) \leq \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y) - z| d\lambda(y) + \varepsilon \to \varepsilon.$$

Taking $\varepsilon \to 0$ we have that $x \in L_f$.

So $L_f^c \subset N$. □

✓ A family $(A_r)_r$ of sets is said to shrink nicely to x if for all $r > 0$, $A_r \subset B(x,r)$ and $\lambda(A_r) \geq \alpha \lambda(B(x,r))$ for some constant α.

For example, if $A \subset B(0,1)$ is a Borel set of positive measure $\alpha = \lambda(A) > 0$, then $rA + x \subset B(x,r)$ and $\lambda(rA + x) = \alpha r^d \geq c\alpha\lambda(B(x,r))$, so $(rA + x)_r$ shrink nicely to x.

●●● **Theorem 15.9** (Lebesgue Differentiation Theorem)**.** *Let $f \in L^1_{\text{loc}}$ and any $x \in L_f$, if $(A_r)_r$ shrink nicely to x then*

$$\lim_{r \to 0} \frac{1}{\lambda(A_r)} \int_{A_r} |f(y) - f(x)| d\lambda(y) = 0.$$

Proof. For some $\alpha > 0$,

$$\frac{1}{\lambda(A_r)} \int_{A_r} |f(y) - f(x)| d\lambda(y) \leq \frac{1}{\alpha\lambda(B(x,r)} \int_{B(x,r)} |f(y) - f(x)| d\lambda(y) \to 0.$$

□

15.3. Differentiation and Radon-Nykodim derivative

We now make a connection between differentiation and the Radon-Nykodim derivative.

A measure μ on $(\mathbb{R}^d, \mathcal{B}_d)$ is called **regular** if $\mu(K) < \infty$ for every compact K and for every Borel A,

$$\mu(A) = \inf_{A \subset U \text{ open}} \mu(U).$$

► **Exercise 15.5.** Let $f : \mathbb{R}^d \to [0, \infty]$ be a measurable function, and define $d\mu = f d\lambda$; *i.e.* $\mu(A) = \int_A f d\mu$.

Show that μ is regular if and only if $f \in L^1_{\text{loc}}$.

♣ **Solution to ex:15.5.** :(

If $f \in L^1_{\text{loc}}$ then since any compact set K is bounded, we have that $\mu(K) = \int_K f d\lambda < \infty$. Also, for any ball B, $f\mathbf{1}_B \in L^1$, and so the measures μ, λ restricted to B are finite. Thus, for any $\varepsilon > 0$ there exists $\delta > 0$ so that if $\lambda(A \cap B) < \delta$ then $\mu(A \cap B) < \varepsilon$. For any Borel $A \subset B$, and any $\delta > 0$ we can find an open set $A \subset U \subset B$ such that $\lambda(U \setminus A) < \delta$. So $\mu(U) - \mu(A) = \mu(U \setminus A) < \varepsilon$. This proves outer regularity for bounded A.

If A is unbounded, write $A = \bigcup_n A_n$ where A_n is bounded for all n. Then, for any $\varepsilon > 0$ take an open set $U_n \supset A_n$ such that $\mu(U_n) \leq \mu(A_n) + \varepsilon 2^{-n}$. So for $U = \bigcup_n U_n$ we have $\mu(U) \leq \mu(A) + \varepsilon$.

For the other direction, if $d\mu = f d\lambda$, then for any bounded Borel set A we may find a compact set K containing A. So $\int_A f d\lambda \leq \int_K f d\lambda = \mu(K) < \infty$. :) ✓

••• **Theorem 15.10.** *Let μ be a regular Borel measure on $(\mathbb{R}^d, \mathcal{B}_d)$. Let $\mu = \sigma + \rho$ be the Lebesgue decomposition with respect to Lebesgue measure λ, so $\sigma \perp \lambda$ and $\rho \ll \lambda$.*

Then, for λ-a.e. $x \in \mathbb{R}^d$,

$$\lim_{r \to 0} \frac{\mu(A_r)}{\lambda(A_r)} = \frac{d\rho}{d\lambda}(x),$$

for any family $(A_r)_r$ that shrinks nicely to x.

Proof. First note that if $x \in L_f$,

$$\lim_{r \to 0} \frac{\rho(A_r)}{\lambda(A_r)} = \lim_{r \to 0} \frac{1}{\lambda(A_r)} \int_{A_r} \frac{d\rho}{d\lambda} d\lambda(y) = \frac{d\rho}{d\lambda}(x),$$

by the Lebesgue Differentiation Theorem.

So we only need to show that for a.e. x,

$$\lim_{r\to 0} \frac{\sigma(A_r)}{\lambda(A_r)} = 0.$$

Note that since $A_r \subset B(x,r)$,

$$\frac{\sigma(A_r)}{\lambda(A_r)} \leq \frac{\sigma(B(x,r))}{\alpha\lambda(B(x,r))},$$

so we may assume that $A_r = B(x,r)$.

Let E be a Borel set such that $\sigma(E) = 0$ and $\lambda(E^c) = 0$. For a positive integer k set

$$F_k = \left\{ x \in E \ : \ \limsup_{r\to 0} \tfrac{\sigma(B(x,r))}{\lambda(B(x,r))} > \tfrac{1}{k} \right\}.$$

It suffices to prove that $\lambda(F_k) = 0$ for all k.

μ is outer regular, so for any $\varepsilon > 0$ there exists an open set $U_\varepsilon \supset E$ such that

$$\sigma(U_\varepsilon) + \lambda(U_\varepsilon) = \mu(U_\varepsilon) \leq \mu(E) + \varepsilon = \lambda(E) + \varepsilon,$$

which implies that $\sigma(U_\varepsilon) \leq \varepsilon$. For every $x \in F_k$ there exists an open ball $B_x \subset U_\varepsilon$ such that $\sigma(B_x) > \frac{1}{k}\lambda(B_x)$. Set $V_\varepsilon = \bigcup_{x\in F_k} B_x \subset U_\varepsilon$. So for any $c < \lambda(V_\varepsilon)$ there exist finitely many pairwise disjoint balls $B_{x_1}, \ldots, B_{x_n}$ such that

$$c < 3^d \sum_{j=1}^n \lambda(B_{x_j}) \leq 3^d \tfrac{1}{k} \sum_{j=1}^n \sigma(B_{x_j}) \leq \frac{3^d}{k} \sigma(V_\varepsilon) \leq \frac{3^d}{k}\varepsilon.$$

So $\sigma(F_k) \leq \frac{3^d}{k}\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we get that $\sigma(F_k) = 0$. □

► **Exercise 15.6.** For $f \in L^1_{\mathrm{loc}}$, define

$$H^* f(x) = \sup\left\{ \tfrac{1}{\lambda(B)} \int_B |f| d\lambda \ : \ B = B(y,r) \ , \ x \in B \right\}.$$

Show that $Hf \leq H^* f \leq 2^d Hf$.

Number of exercises in lecture: 6

Total number of exercises until here: 172

Measure Theory

Ariel Yadin

Lecture 16: The Riesz Representation Theorem

16.1. Compactly supported functions

Let X be some topological space. Let $C_c(X)$ denote the space of all functions $f : X \to \mathbb{C}$ that have $\mathrm{supp}(f) = \mathrm{cl}(\{x : f(x) \neq 0\})$ is a compact set.

We will use the notation $f \prec U$ to denote $f : X \to [0,1]$, $\mathrm{supp}(f) \subset U$, $f \in C_c(X)$.

• **Definition 16.1.** We say that a space X is **Urysohn** if for any $x \in U$ where U is open, there exists $f \prec U$ and an open set V such that $x \in V \subset \mathrm{supp}(f)$ with $\mathbf{1}_V \leq f$.

Example 16.2. Any locally compact Hausdorff space is Urysohn. (This is a consequence of what is known as Urysohn's Lemma.)

The notion of locally compact Hausdorff spaces is an important one in topology, but it is beyond the scope of this course. △ ▽ △

► **Exercise 16.1.** Show that $\mathbb{R}^d$ is Urysohn.

♣ **Solution to ex:16.1.** :(

Let $x \in U$ for U open. So there exists an open ball $B = B(x, 3\varepsilon)$ such that $x \in B(x, 3\varepsilon) \subset U$.

Define

$$f(y) = \begin{cases} 1 & \text{if } ||y - x|| \leq \varepsilon, \\ 2 - \varepsilon^{-1} \cdot ||y - x|| & \text{if } \varepsilon \leq ||y - x|| \leq 2\varepsilon, \\ 0 & \text{if } ||y - x|| \geq 2\varepsilon. \end{cases}$$

Since the function $y \mapsto ||y - x||$ is continuous, so is f.

Since $x \in B(x,\varepsilon) \subset \mathrm{cl}B(x,2\varepsilon) = \mathrm{supp}(f) \subset U$ and $f\big|_{B(x,\varepsilon)} \equiv 1$, we are done. :) ✓

► **Exercise 16.2.** Let X be a countable set. Consider X as a topological space with the discrete topology (*i.e.* all subsets are open; alternatively, consider the metric $\mathrm{dist}(x,y) = 0$ for $x = y$ and $\mathrm{dist}(x,y) = 1$ for $x \neq y$.

Show that a subset $K \subset X$ is compact if and only if K is finite.

Show that X is Urysohn.

♣ **Solution to ex:16.2.** :(

If K is compact then $K \subset \bigcup_{x\in K} \{x\}$ is an open cover, so there exists a finite sub-cover, that is $K \subset \bigcup_{j=1}^n \{x_j\}$ for some $x_1,\ldots,x_n \in K$. This implies that $K = \{x_1,\ldots,x_n\}$ is finite.

For the other direction, if K is finite and $K \subset \bigcup_\alpha U_\alpha$ is an open cover, then for every $x \in K$ there exists $\alpha = \alpha(x)$ such that $x \in U_x := U_\alpha$. So $K \subset \bigcup_{x\in K} U_x$ is a finite sub-cover. Every open cover has a finite sub-cover, so K is compact.

We now show that X is Urysohn. If $x \in U$ for open U, then define $f = \mathbf{1}_{\{x\}}$. Since $\{x\}$ is open and compact we have that $x \in \{x\} \subset \mathrm{supp}(f) \subset U$. Finally, f is continuous since for the discrete topology any function is continuous. :) ✓

► **Exercise 16.3.** Show that if $f \in C_c(X)$ is a non-negative function then $g := 1 \wedge f$ is compactly supported and continuous; $g \in C_c(X)$, $g : X \to [0,1]$.

♣ **Solution to ex:16.3.** :(

It is immediate that $\mathrm{supp}(g) \subset \mathrm{supp}(f)$, so we only need to show that g is continuous.

Two proofs:

Proof I. If $U \subset \mathbb{R}$ is an open set, then so is $V = U \setminus \{1\}$. If $1 \notin U$ then $g^{-1}(U) = f^{-1}(U)$ which is open. If $1 \in U$ then $g(x) \in U$ if and only if either $f(x) \in U$ or $f(x) > 1$. So $g^{-1}(U) = f^{-1}(U) \bigcup \{f > 1\}$ which is also open.

Proof II. Let $x_n \to x$. If $f(x) < 1$ then because f is continuous, there exists n_0 such that for all $n > n_0$ we have $f(x_n) < 1$. Thus, $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_n) = f(x) = g(x)$. If $f(x) \geq 1$ then for any $\varepsilon > 0$ there exists n_0 such that for all $n > n_0$ we have $|f(x_n) - f(x)| < \varepsilon$, which implies that $f(x_n) \geq 1 - \varepsilon$. This implies that $g(x_n) \geq 1 - \varepsilon$ for all $n > n_0$. So $\liminf_n g(x_n) \geq 1 - \varepsilon$. Since this holds for all $\varepsilon > 0$, we have that $\limsup_n g(x_n) \leq 1 \leq \liminf_n g(x_n)$ so $\lim_{n\to\infty} g(x_n) = 1 = g(x)$. :) ✓

• **Proposition 16.3.** *Let X be Urysohn.*

Let $K \subset U$ for compact K and open U. Then there exists $f \in C_c(X)$ such that $\mathbf{1}_K \leq f \prec U$.

Proof. For any $x \in K \subset U$ we can find an open set V_x and a function $f_x \prec U$ such that $x \in V_x \subset \text{supp}(f_x)$ and $\mathbf{1}_{V_x} \leq f$.

Since K is compact and $K \subset \bigcup_{x\in K} V_x$ is an open cover, we have a finite sub-cover; that is, there exist $x_1, \ldots, x_m$ such that $K \subset \bigcup_{j=1}^m V_{x_j}$.

Set $f := \sum_{j=1}^m f_{x_j}$. Then $\mathbf{1}_K \leq \sum_{j=1}^m \mathbf{1}_{V_{x_j}} \leq f$ and $\text{supp}(f) \subset \bigcup_{j=1}^n \text{supp}(f_{x_j}) \subset U$. So $f \in C_c(X)$ and non-negative. Thus, $1 \wedge f \prec U$ and $\mathbf{1}_K \leq 1 \wedge f$. □

• **Proposition 16.4.** *Let X be an Urysohn space. Let $K \subset \bigcup_{j=1}^n U_j$ for compact K and open $(U_j)_{j=1}^n$. Then, there exist functions $f_j \prec U_j$ such that $\sum_{j=1}^n f_j(x) = 1$ for all $x \in K$.*

Proof. For every $x \in K$ there exists j such that $x \in U_j$. So we can find a function $g_x \prec U_j$ and an open set V_x such that $x \in V_x \subset \text{supp}(g_x) \subset U_j$ and $\mathbf{1}_{V_x} \leq g_x$.

Since K is compact and $K \subset \bigcup_{x\in K} V_x$ is an open cover, we have that there exists a finite sub-cover; *i.e.* there exist $x_1, \ldots, x_m \in K$ such that $K \subset \bigcup_{i=1}^m V_{x_i} \subset \bigcup_{i=1}^m \text{supp}(g_{x_i})$. For every j set

$$K_j = \bigcup \{\text{supp}(g_{x_i}) \ : \ \text{supp}(g_{x_i}) \subset U_j\} .$$

Then K_j is a finite union of compact sets, so it is compact. Also, by definition, $K_j \subset U_j$.

Thus, we may find a function $g_j \in C_c(X)$ such that $\mathbf{1}_{K_j} \leq g_j \prec U_j$.

Let $g = \sum_{j=1}^n g_j$. Since for every x_i there exists j such that $\mathrm{supp}(g_{x_i}) \subset U_j$, then $K \subset \bigcup_{i=1}^m \mathrm{supp}(g_{x_i}) \subset \bigcup_{j=1}^n K_j$. So for all $x \in K$ we have that $g(x) \geq 1$. Let $V = \{g > 0\}$ which is an open set because g is continuous. Since $K \subset V$, we can find $h \in C_c(X)$ such that $\mathbf{1}_K \leq h \prec V$.

Define $f = g + 1 - h$. Since $\mathrm{supp}(h) \subset \{g > 0\}$, we have that $f(x) > 0$ for all x. Thus, we can define $f_j := \frac{g_j}{f}$.

For $x \in K$, since $h(x) = 1$, then $\sum_{j=1}^n f_j(x) = \sum_{j=1}^n \frac{g_j(x)}{g(x)} = 1$. Also, $\mathrm{supp}(f_j) \subset \mathrm{supp}(g_j) \subset U$ and so $f_j \prec U$. □

16.2. Linear functionals

• **Definition 16.5.** Let F be some space of functions $f : X \to \mathbb{C}$. A **linear functional** I on F is a function $I : F \to \mathbb{C}$ such that $I(\alpha f + g) = \alpha I(f) + I(g)$ for all $\alpha \in \mathbb{C}, f, g \in F$.

A linear functional is called **positive** if for any $f \in F$ that is non-negative, we have $I(f) \geq 0$.

✓ Recall that $f \geq 0$ means that f is non-negative. Also, $f \geq g$ means that $f - g \geq 0$.

▶ **Exercise 16.4.** Let $(X, \mathcal{F}, \mu)$ be a measure space. Show that the function $I(f) = \int f d\mu$ is a positive linear functional on $L^1(X, \mathcal{F}, \mu)$.

• **Proposition 16.6.** *Let X be an Urysohn topological space. Let I be a positive linear functional on $C_c(X)$. Then, for any compact $K \subset X$ there exists a constant $C_K > 0$ such that for any $f \in C_c(X)$ with $\mathrm{supp}(f) \subset K$ we have $I(f) \leq C_K \cdot ||f||_\infty$ (recall $||f||_\infty = \sup_{x \in X} |f(x)|$).*

Proof. For general $f \in C_c(X)$, since $f = \mathrm{Re} f + i \mathrm{Im} f$ we have $|I(f)| \leq |I(\mathrm{Re} f)| + |I(\mathrm{Im} f)|$, so it suffices to consider real-valued f.

Given a compact K, Let $\varphi_K \in C_c(X)$ be a function $\varphi_K : X \to [0,1]$ such that $\varphi\big|_K \equiv 1$ (X is always open, so this is possible by Urysohn).

If $\text{supp}(f) \subset K$ then $|f(x)| \leq \varphi_K(x) \cdot ||f||_\infty$ for any $x \in X$. So $||f||_\infty \cdot \varphi_K - f \geq 0$ and $||f||_\infty \cdot \varphi_K - (-f) \geq 0$ $||f||_\infty \cdot I(\varphi_K) - I(f) \geq 0$ and $||f||_\infty \cdot I(\varphi_K) + I(f) \geq 0$, which is $|I(f)| \leq I(\varphi_K) \cdot ||f||_\infty$. □

16.3. The Riesz Representation Theorem

• **Definition 16.7.** Let X be a topological space. A **Borel** measure on X is a measure on $(X, \mathcal{B}(X))$.

A Borel measure μ is called **Radon** if

- For any compact $K \subset X$ we have $\mu(K) < \infty$.
- μ is outer regular; *i.e.* for all Borel sets A,

$$\mu(A) = \inf_{A \subset U \text{ open}} \mu(U).$$

- μ is inner regular on open sets; *i.e.* for any open set U,

$$\mu(U) = \sup_{U \supset K \text{ compact}} \mu(K).$$

✓ The notation $f \prec U$ is short for $f \in C_c(X)$, $f : X \to [0,1]$ and $\text{supp}(f) \subset U$.

★★★ **THEOREM 16.8** (Riesz Representation Theorem)**.** *Let X be an Urysohn space. Let I be a positive linear functional on $C_c(X)$. Then, there exists a* unique *Radon measure μ on X such that $I(f) = \int f d\mu$ for all $f \in C_c(X)$.*

Moreover, the measure μ satisfies

$$\mu(U) = \sup \{I(f) \ : \ f \prec U\} \qquad \forall \ \text{open } U,$$

$$\mu(K) = \inf \{I(f) \ : \ f \in C_c(X) \ , \ f \geq \mathbf{1}_K\} \qquad \forall \ \text{compact } K.$$

Proof. The statement of the theorem itself suggest how to define μ: Define

$$\mu(U) := \sup \{I(f) \ : \ f \prec U\}$$

for any open $U \subset X$.

Step I. μ is countably sub-additive on open sets; *i.e.* if $U = \bigcup_n U_n$ for open sets $(U_n)_n$, we claim that $\mu(U) \leq \sum_n \mu(U_n)$.

Indeed, for any $f \in C_c(X)$ such that $f \prec U$, we have that $K = \mathrm{supp}(f)$ is compact and $K \subset \bigcup_n U_n$ is an open cover. So there exists a finite sub-cover $K \subset \bigcup_{j=1}^n U_j$. We may now find $g_1, \ldots, g_n$ such that $g_j \prec U_j$ and $\sum_{j=1}^n g_j(x) = 1$ for all $x \in K$. This implies that $f = \sum_{j=1}^n f g_j$ and $f g_j \prec U_j$. So by definition of μ,

$$I(f) = \sum_{j=1}^n I(f g_j) \leq \sum_{j=1}^n \mu(U_j) \leq \sum_n \mu(U_n).$$

Since this holds for any $f \prec U$, we get that $\mu(U) \leq \sum_n \mu(U_n)$.

For an arbitrary subset $A \subset X$ define

$$\mu^*(A) = \inf_{A \subset U \text{ open}} \mu(U).$$

(*) Exercise: Show that $\mu^*(U) = \mu(U)$ for all open U.

Step II. μ^* is an outer measure.

Recall the axioms of an outer measure: $\mu^*(\emptyset) = 0$, $\mu^*(A) \leq \mu^*(B)$ for all $A \subset B$ and $\mu^*(\bigcup_n A_n) \leq \sum_n \mu^*(A_n)$.

The first two are easy. Now for the third axiom.

Since a countable union of open sets is open, we have that

$$\mu^*(A) = \inf \left\{ \sum_n \mu(U_n) \ : \ A \subset \bigcup_n U_n \ , \ U_n \text{ are all open} \right\}.$$

(*) Exercise: prove this.

Let $A = \bigcup_n A_n$. If $\mu^*(A_n) = \infty$ for some n there is nothing to prove. So assume that $\mu^*(A_n) < \infty$ for all n. Fix $\varepsilon > 0$ and for every n let U_n be an open set such that $A_n \subset U_n$ and $\mu(U_n) \leq \mu^*(A_n) + \varepsilon \cdot 2^{-n}$. Then, $A \subset \bigcup_n U_n$ and

$$\mu^*(A) \leq \sum_n \mu(U_n) \leq \sum_n \mu^*(A_n) + \varepsilon.$$

Taking $\varepsilon \to 0$ completes the proof that μ^* is an outer measure (step II).

Step III. Every open set U is μ^*-measurable.

Recall that U is μ^*-measurable if and only if for any set A such that $\mu^*(A) < \infty$, we have $\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \cap U^c)$. If A is such a set, fix some $\varepsilon > 0$ and let V be an open set such that $A \subset V$ and $\mu(V) \leq \mu^*(A) + \varepsilon$. Since $V \cap U$ is open, by definition of μ there exists $f \in C_c(X)$, $f \prec V \cap U$, such that $I(f) \geq \mu(V \cap U) - \varepsilon$. Also, the set $K = \text{supp}(f)$ is closed so $V \cap K^c$ is open. So we may find $g \in C_c(X)$, $g \prec V \cap K^c$, such that $I(g) \geq \mu(V \cap K^c) - \varepsilon$. Now note that $\text{supp}(f) \cap \text{supp}(g) = \emptyset$ because $\text{supp}(g) \subset K^c$. So $0 \leq f + g \leq 1$ and $\text{supp}(f+g) \subset K \cup (V \cap K^c) \subset V$. Thus, by definition of μ,

$$\begin{aligned}\mu^*(A) &\geq \mu(V) - \varepsilon \geq I(f+g) - \varepsilon = I(f) + I(g) - \varepsilon \\ &\geq \mu(V \cap U) + \mu(V \cap K^c) - 3\varepsilon \geq \mu(A \cap U) + \mu(A \cap U^c) - 3\varepsilon.\end{aligned}$$

Taking $\varepsilon \to 0$ completes the proof that U is μ^*-measurable (step III).

Step IV. By Charathéodory's Theorem μ^* restricted to the Borel sets is a measure. Denote this measure by μ.

For any Borel set A we have a sequence of open sets $(U_n)_n$ such that $\mu(U_n) \to \mu(A)$ and $A \subset U_n$ for all n. So μ is outer-regular.

Step V. We show that $\mu(K) = \inf \{I(f) \ : \ f \geq \mathbf{1}_K\}$ for all compact K.

If K is compact and $f \geq \mathbf{1}_K$ then for any $\varepsilon > 0$ let $U_\varepsilon = \{f > 1 - \varepsilon\}$. Then U_ε is open (because f is continuous). For any $g \prec U_\varepsilon$ we have that $f > (1-\varepsilon)g$, so $I(f) \geq (1-\varepsilon)I(g)$. Taking supremum over all $g \prec U_\varepsilon$ we have that $\mu(K) \leq \mu(U_\varepsilon) \leq (1-\varepsilon)^{-1} I(f)$. Since this holds for all ε, taking $\varepsilon \to 0$ we get that $\mu(K) \leq I(f)$. This was for any $f \geq \mathbf{1}_K$, so $\mu(K) \leq \inf \{I(f) \ : \ f \geq \mathbf{1}_K\}$

For the other inequality, let $U \supset K$ be any open set. Then we can find $f_U \in C_c(X)$, such that $f\big|_K \equiv 1$ and $f_U \prec U$. So $f_U \geq \mathbf{1}_K$ and $I(f_U) \leq \mu(U)$. Taking infimum, we have that

$$\inf \{I(f) \ : \ f \geq \mathbf{1}_K\} \leq \inf \{I(f_U) \ : \ K \subset U \text{ open }\} \leq \inf \{\mu(U) \ : \ K \subset U \text{ open }\} = \mu(K).$$

This proves $\mu(K) = \inf \{I(f) \ : \ f \geq \mathbf{1}_K\}$ for all compact K (step V).

Step VI. $\mu(K) < \infty$ for all compact K. Indeed, there exists $f \in C_c(X)$ such that $f : X \to [0,1]$ and $f\big|_K \equiv 1$. So $f \geq \mathbf{1}_K$, and $I(f) \leq C_K \cdot ||f||_\infty = C_K$ for some constant $C_K > 0$. Thus, $\mu(K) \leq I(f) < \infty$.

Step VII. μ is inner regular on open sets. Indeed, if U is an open set, for any $\alpha < \mu(U)$ let $f \in C_c(X)$, $f \prec U$ be such that $I(f) \geq \alpha$. Set $K := \operatorname{supp}(f)$ which is compact and $K \subset U$.

For any $g \in C_c(X)$ such that $g \geq \mathbf{1}_K$ we have that $g - f \geq 0$, so $I(g) \geq I(f) \geq \alpha$. Taking infimum over all such g we have that $\mu(K) \geq \alpha$.

Thus, for any $\alpha < \mu(U)$ we have a compact $K \subset U$ such that $\mu(K) \geq \alpha$. Taking supremum over α gives inner regularity (step VII).

Step VIII. We now show that $I(f) = \int f d\mu$ for all $f \in C_c(X)$.

If $f \in C_c(X)$ write $f = f_1 - f_2 + i(f_3 - f_4)$ for non-negative f_j, so it suffices to consider non-negative f. Since $f \in C_c(X)$ is continuous on a compact set, namely $\operatorname{supp}(f)$, it attains a maximum there. So $||f||_\infty < \infty$. So replacing f by $\frac{1}{||f||_\infty} f$, we may assume without loss of generality that $f : X \to [0,1]$.

Fix $n > 0$. For any $1 \leq j \leq n$ set $K_j = \{f \geq j/n\}$ and $K_0 = \operatorname{supp}(f)$. For $1 \leq j \leq n$ define

$$f_j = \frac{1}{n} \wedge \left(f - \tfrac{j-1}{n} \right)^+ .$$

That is,

$$f_j(x) = \begin{cases} 0 & f(x) \leq \frac{j-1}{n}, \\ f(x) - \frac{j-1}{n} & f(x) \in [\frac{j-1}{n}, \frac{j}{n}]. \\ \frac{1}{n} & f(x) \geq \frac{j}{n}. \end{cases}$$

With this definition $\frac{1}{n}\mathbf{1}_{K_j} \leq f_j \leq \frac{1}{n}\mathbf{1}_{K_{j-1}}$, so $\frac{1}{n}\mu(K_j) \leq \int f_j d\mu \leq \frac{1}{n}\mu(K_{j-1})$.

Note that f_j is continuous, and $\operatorname{supp}(f_j) \subset K_{j-1}$, so $f_j \in C_c(X)$ (this includes the case $j = 1$ where $K_0 = \operatorname{supp}(f)$).

For any open set $U \supset K_{j-1}$, we have that $nf_j \prec U$. So $nI(f_j) \leq \mu(U)$. Taking infimum over all such open sets $U \supset K_{j-1}$, by outer regularity we have that $I(f_j) \leq \frac{1}{n}\mu(K_{j-1})$.

Also, $nf_j \geq \mathbf{1}_{K_j}$, so $\frac{1}{n}\mu(K_j) \leq I(f_j)$.

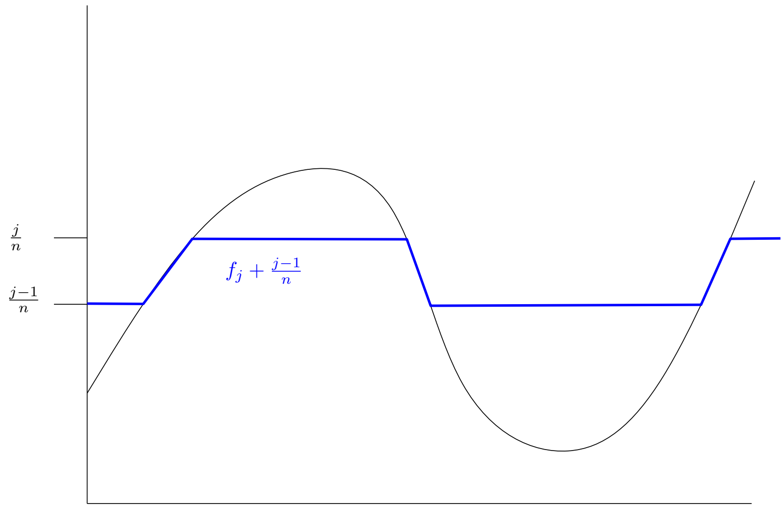

FIGURE 5. The function f_j raised by $\frac{j-1}{n}$.

Now, note that $f = \sum_{j=1}^{n} f_j$. (Exercise!) So by additivity of the integral and the linear functional,

$$\frac{1}{n}\sum_{j=1}^{n}\mu(K_j) \leq I(f) \leq \frac{1}{n}\sum_{j=1}^{n}\mu(K_{j-1}) \qquad \text{and}$$
$$\frac{1}{n}\sum_{j=1}^{n}\mu(K_j) \leq \int f d\mu \leq \frac{1}{n}\sum_{j=1}^{n}\mu(K_{j-1}).$$

Taking the differences between these inequalities,

$$\left|I(f) - \int f d\mu\right| \leq \frac{1}{n}\sum_{j=1}^{n}(\mu(K_{j-1}) - \mu(K_j)) = \frac{\mu(K_0) - \mu(K_n)}{n} \leq \frac{1}{n}\mu(\mathrm{supp}(f)).$$

μ is finite on compact sets (as a Radon measure), so $\mu(\mathrm{supp}(f)) < \infty$. Thus, taking $n \to \infty$ we have that $I(f) = \int f d\mu$ as required (step VIII).

Step IX. Uniqueness. Let ν be another Radon measure satisfying $I(f) = \int f d\nu$ for all $f \in C_c(X)$. For any open set U, if $f \prec U$ then $I(f) = \int_U f d\nu \leq \nu(U)$. This holds for any $f \prec U$ so

$$\nu(U) \geq \sup\{I(f) \ : \ f \prec U\}.$$

Since ν is inner regular on open sets, we can find a sequence of compact sets $(K_n)_n$ such that $K_n \subset U$ and $\nu(K_n) \to \nu(U)$. For every n, Urysohn guaranties that there exists a function $f_n \in C_c(X)$ such that $f_n\big|_{K_n} \equiv 1$ and $f_n \prec U$. So

$$\nu(K_n) \leq \int_{K_n} f_n d\nu \leq \int f_n d\nu = I(f_n) \leq \sup\{I(f) \ : \ f \prec U\}.$$

Taking $n \to \infty$ we get that $\nu(U) = \sup\{I(f) \ : \ f \prec U\} = \mu(U)$.

So $\nu(U) = \mu(U)$ for all open sets U. Since ν, μ are outer regular, this implies that they are equal on all Borel sets. □

► **Exercise 16.5.** Let X be an Urysohn space. Let $F \subset X$ be a closed subset. Let μ be a Radon measure on F. Define $I(f) = \int f\big|_F d\mu$ for all $f \in C_c(X)$.

Show that I is a positive linear functional on $C_c(X)$.

Show that if $I(f) = \int f d\nu$ for the Radon measure ν guarantied by the Riesz Representation Theorem, then $\nu(A) = \mu(A \cap F)$ for all Borel A.

Number of exercises in lecture: 5

Total number of exercises until here: 177