Percolation - 201.2.0101

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Home Exam Spring 2013

Due date: August 5, 2013

By this date it should be in my box at BGU.

It is a good idea to send me an email when you have submitted your work.

Instructions

- You may use any resources you find such as lecture notes, books, the internet. If you use solutions of exercises found in such, you should reference them in your solution. (For example, "*in my solution I use a technique from: G. Orwell. Animal Farm. Chapter 5.*".) If you use such resources, you should not copy a solution, but rather understand it and write it up in your own words.
- Work in pairs is permitted, but only in pairs, not more. Do not write up the solutions together, but rather after understanding the solution, each of you should write them up on your own.
- You may use any theorems proved in class, but any other propositions you wish to use - you should prove.
- Please write clearly. State your claims and proofs in a mathematical format, with precise notation and correct formalism.
- Make sure your ID no. appears on your solutions.
- Each exercise is worth 20 points, although the different parts of a specific exercise are not all equal. Extra points will be awarded for original ideas and proofs.
- By taking this exam, you agree to these terms, and declare that you will conform to them.

Exercise 1. *Regarding the number of infinite components in percolation:*

(A) *Let G be an infinite connected graph (not necessarily transitive). Let $p \in [0,1]$. Show that for p-percolation on $G \times \mathbb{Z}$ the number of infinite components is constant a.s. and is either $0, 1$ or ∞.*

(Recall that the graph $G \times \mathbb{Z}$ has vertex set $V(G) \times \mathbb{Z}$ and edges given by $(x,k) \sim (y,k)$ if and only if $x \sim y$ and $(x,k) \sim (x,k+1)$.)

(B) *Give an example of a connected bounded degree graph G such that the number of infinite components in p-percolation on G is not a constant (that is, show that for some k, $0 < \mathbb{P}_p[N = k] < 1$, where N is the random variable that is the number of infinite components).*

Solution to Exercise 1.

(A) For $z \in \mathbb{Z}$ consider the automorphism $\varphi_z(g,k) = (g, k+z)$.

Let N be the number of infinite components and $A = \{N = k\}$. First we show that $\mathbb{P}_p[A] \in \{0,1\}$.

Indeed, note that $\mathbb{P}_p[\varphi_z B] = \mathbb{P}_p[B]$ for any event B, because $\varphi_z \Omega$ has the same distribution as percolation on $G \times (\mathbb{Z} + z)$, an isomorphic copy of $G \times \mathbb{Z}$.

Also, since the number of infinite components does not change under φ_z, we have that $\varphi_z A = A$.

Finally, for any finite set $E \subset G \times \mathbb{Z}$ we have that there exists $z \in \mathbb{Z}$ such that $\varphi_z E \cap E = \emptyset$.

Now, let $(A_n)_n$ be a sequence of events such that $\mathbb{P}[A \triangle A_n] \to 0$ and $A_n \in \mathcal{F}_{E_n}$ for some finite set E_n. Let $\varphi_n = \varphi_{z_n}$ for z_n such that $\varphi_{z_n} E_n \cap E_n = \emptyset$.

For any $z \in \mathbb{Z}$, since $A = \varphi_z A$, we have that

$$\begin{aligned}\mathbb{P}[A] - \mathbb{P}[A_n \cap \varphi_z A_n] &\leq \mathbb{P}[A \setminus (A_n \cap \varphi_z A_n)] \leq \mathbb{P}[A \setminus A_n] + \mathbb{P}[A \setminus \varphi_z A_n] \\ &= \mathbb{P}[A \setminus A_n] + \mathbb{P}[\varphi_z (A \setminus A_n)] = 2\,\mathbb{P}[A \setminus A_n].\end{aligned}$$

So, because A_n is independent of $\varphi_n A_n$,

$$\mathbb{P}[A] \leq \mathbb{P}[A_n]\,\mathbb{P}[\varphi_n A_n] + 2\,\mathbb{P}[A \setminus A_n] = \mathbb{P}[A_n]^2 + 2\,\mathbb{P}[A \setminus A_n] \to \mathbb{P}[A]^2 \leq \mathbb{P}[A].$$

So $\mathbb{P}[A] = \mathbb{P}[A]^2$ and $\mathbb{P}[A] \in \{0,1\}$.

This process that for all $p \in [0,1]$ there exists k_p such that $\mathbb{P}[N = k_p] = 1$.

Let B be a ball in $G \times \mathbb{Z}$. $N_{0,B}$ be the number of infinite components when the sites in B are forced to be closed, and let $N_{1,B}$ be the number of infinite components when the sites in B are forced to be open. If O_B is the event that B is open and C_B is the event that B is closed then because $N_{1,B}, N_{0,B}$ are independent of $\mathcal{F}_B$ and $O_B, C_B \in \mathcal{F}_B$,

$$\mathbb{P}[N_{1,B} = k_p] = \mathbb{P}[N_{1,B} = k_p \mid O_B] = \mathbb{P}[N_{1,B} = N = k_p] = 1,$$

$$\mathbb{P}[N_{0,B} = k_p] = \mathbb{P}[N_{0,B} = k_p \mid C_B] = \mathbb{P}[N_{0,B} = N = k_p] = 1.$$

Let N_B be the number of infinite components intersecting B. $N_{0,B} \geq N_B$, because closing the site in B can only disconnect components. Opening the site in B connects all components intersecting B, so if $N_B \geq 2$ and $N < \infty$ then $N_{1,B} \leq N_B - 1$. So, if $k_p < \infty$ then

$$\mathbb{P}[N_B \geq 2] = \mathbb{P}[N_B \geq 2, N < \infty] \leq \mathbb{P}[N_{0,B} \geq N_B, N_{1,B} \leq N_B - 1] \leq \mathbb{P}[N_{0,B} \neq N_{1,B}] = 0.$$

Now, letting $N_r := N_{B(o,r)}$ for some o, we have that $N_r \nearrow N$. So if $k_p < \infty$,

$$\mathbb{P}[N \geq 2] = \lim_{r \to \infty} \mathbb{P}[N_r \geq 2] = 0.$$

(B) Fix p and let G, G' be two copies of a transitive graph with $p > p_u(G)$. Let $o \in G$ and let $o' \in G'$ be the copy of o in G'. Define the graph Γ be taking $G \cup G'$ and adding an edge between o and o'.

If we are in bond percolation, then if the edge $o \sim o'$ is closed and both o and o' are in an infinite component of G and G' respectively, then we have at least two infinite components. This has probability $(1-p)\theta(p)^2 > 0$.

If $o \sim o'$ is open and o and o' are both in an infinite component in G and G' respectively, then because we are in the uniqueness phase in each of G, G', we have that there is exactly one infinite component in Γ. This happens with probability $p\theta(p)^2 > 0$.

So if N is the number of infinite components,

$$0 < p\theta(p)^2 \leq \mathbb{P}[N = 1] \leq 1 - \mathbb{P}[N = 2] \leq 1 - (1-p)\theta(p)^2 < 1.$$

□

Exercise 2. *Let G be a transitive infinite connected graph. Consider bond percolation on G. Let $p < p_c$.*

(A) *An auxiliary claim: Show that if Γ is a* finite *connected graph, and $x, y, z \in \Gamma$ three distinct vertices, then there exists a vertex $v \in \Gamma$ such that there exist three* edge disjoint *paths in Γ, $\alpha : x \to v$, $\beta : y \to v$ and $\gamma : z \to v$ (it may be that $v \in \{x, y, z\}$ in which case one may choose an empty path). [*Hint: *A spanning tree of Γ is a connected subgraph T of Γ such that T is a tree and $V(T) = V(\Gamma)$. Every finite connected graph has at least one spanning tree.]*

(B) *Consider p-bond percolation on G, with $p < p_c$, if x, y, z are three vertices (*not *necessarily distinct) such that x, y, z are all in the same component of p-bond percolation on G, then there exists a vertex $v \in G$ such that $\{v \leftrightarrow x\} \circ \{v \leftrightarrow y\} \circ \{v \leftrightarrow z\}$.*

(C) *Use the above to prove that*

$$\mathbb{E}_p[|\mathcal{C}|^2] \leq \left(\mathbb{E}_p[|\mathcal{C}|]\right)^3.$$

Solution to Exercise 2.

(A) Let T be a spanning tree of Γ. Let $\delta : x \to y$ be the unique simple path in T connecting x to y in T. Suppose that $\delta = (\delta_0, \ldots, \delta_n)$. ($T$ is a tree). There is a unique simple path $\eta : z \to x$ in T connecting z to x. Suppose that $\eta = (\eta_0, \dots, \eta_k)$. Set $J = \min\{j : \eta_j \in \delta\}$; that is η_J is the first vertex in η that is also in δ. This must exist since $\eta_k = x = \delta_0$. Since δ, η are simple paths, we can choose I such that $\delta_I = \eta_J$ is the first intersection point, and set $v := \delta_I = \eta_J$.

We have that $\alpha := (\delta_0, \dots, \delta_I)$, $\beta := (\delta_n, \delta_{n-1}, \dots, \delta_I)$ and $\gamma := (\eta_0, \dots, \eta_J)$ are all edge disjoint, and $\alpha : x \to v$, $\beta : y \to v$ and $\gamma : z \to v$, as required.

(B) First consider the case where x, y, z are all distinct. Since $p < p_c$, the graph $\Gamma = \mathcal{C}(x)$ is a finite connected graph. Thus, by (A), since $x, y, z \in \mathcal{C}(x)$ we can find $v \in \mathcal{C}(x)$ and edge disjoint paths α, β, γ in $\mathcal{C}(x)$ such that $\alpha : x \to v, \beta : y \to v, \gamma : z \to v$. These paths are in $\mathcal{C}(x)$, so specifically are open paths. That is we have the event, $\{v \leftrightarrow x\} \circ \{v \leftrightarrow y\} \circ \{v \leftrightarrow z\}$.

Now, if x, y, z are not all distinct, then there are two cases: Either $x = y = z$ in which case taking the empty path from x to itself and $v = x = y = z$ gives $\{x \leftrightarrow x\} \circ \{y \leftrightarrow y\} \circ \{z \leftrightarrow z\} = \{v \leftrightarrow x\} \circ \{v \leftrightarrow y\} \circ \{v \leftrightarrow z\}$. In the other case, without loss of generality, $x = y \neq z$. Then we choose $v = x = y$ and any open path $\gamma : z \to v$ (since $z \leftrightarrow x$), and take α, β to be the empty path. This again gives $\{x \leftrightarrow x\} \circ \{y \leftrightarrow y\} \circ \{z \leftrightarrow x\} = \{v \leftrightarrow x\} \circ \{v \leftrightarrow y\} \circ \{v \leftrightarrow z\}$.

(C) G is transitive, so we may take $\mathcal{C} = \mathcal{C}(x)$. Fix some large $r > 0$. Let $\mathcal{C}_r = \mathcal{C} \cap B(x, r)$. We start with

$$\mathbb{E}_p[|\mathcal{C}_r|^2] = \sum_{y,z \in B(x,r)} \mathbb{E}_p[\mathbf{1}_{\{y \in \mathcal{C}_r, z \in \mathcal{C}_r\}}] = \sum_{y,z \in B(x,r)} \mathbb{P}_p[x, y, z \in \mathcal{C}_r].$$

By (B) the event $\{x, y, z \in \mathcal{C}_r\}$ implies that there exists $v \in B(x, r)$ such that $\{v \leftrightarrow x\} \circ \{v \leftrightarrow y\} \circ \{v \leftrightarrow z\}$. That is, by the BK-inequality

$$\mathbb{P}_p[x, y, z \in \mathcal{C}_r] \leq \sum_{v \in B(x,r)} \mathbb{P}[\{v \leftrightarrow x\} \circ \{v \leftrightarrow y\} \circ \{v \leftrightarrow z\}] \leq \sum_{v \in B(x,r)} \mathbb{P}_p[v \leftrightarrow x]\, \mathbb{P}_p[v \leftrightarrow y]\, \mathbb{P}_p[v \leftrightarrow z].$$

So

$$
\begin{aligned}
\mathbb{E}_p[|\mathcal{C}_r|^2] &\leq \sum_{y,z\in B(x,r)} \sum_{v\in B(x,r)} \mathbb{P}_p[v \leftrightarrow x]\, \mathbb{P}_p[v \leftrightarrow y]\, \mathbb{P}_p[v \leftrightarrow z] \\
&= \sum_{v\in B(x,r)} \mathbb{P}_p[x \leftrightarrow v] \sum_{y\in B(x,r)} \mathbb{P}_p[v \leftrightarrow y] \sum_{z\in B(x,r)} \mathbb{P}_p[v \leftrightarrow z] \\
&\leq \sum_{v\in G} \mathbb{P}_p[x \leftrightarrow v] \sum_{y\in G} \mathbb{P}_p[v \leftrightarrow y] \sum_{z\in G} \mathbb{P}_p[v \leftrightarrow z].
\end{aligned}
$$

Since G is transitive, we know that for any u,

$$\sum_{w\in G} \mathbb{P}_p[u \leftrightarrow w] = \mathbb{E}_p[|\mathcal{C}|].$$

So for any r,

$$\mathbb{E}_p[|\mathcal{C}_r|^2] \leq \left(\mathbb{E}_p[|\mathcal{C}|]\right)^3.$$

Since $|\mathcal{C}_r|^2 \nearrow |\mathcal{C}|^2$ as $r \to \infty$, we have by monotone convergence that

$$\mathbb{E}_p[|\mathcal{C}|^2] = \lim_{r\to\infty} \mathbb{E}_p[|\mathcal{C}_r|^2] \leq \left(\mathbb{E}_p[|\mathcal{C}|]\right)^3.$$

$\square$

Exercise 3. *Give an example of an infinite connected graph G with bounded degrees and a vertex $o \in G$ such that $p_c(G) = 1$ and such that G has exponential volume growth;* i.e. *there exists constant $c > 1$ and $o \in G$ such that for all $r > 0$, $|B(o,r)| \geq c^r$.*

Show this for both site and bond percolation.

Solution to Exercise 3. Consider the following tree.

For every $k \in \mathbb{N}$ let T_k be a finite binary tree of depth k, rooted at a root vertex o_k. So $|T_k| = 2^0 + 2^1 + \cdots + 2^k = 2^{k+1} - 1$, and the maximal distance in T_k is k.

Consider the graph $\mathbb{N}$ (a subgraph of $\mathbb{Z}$). For every $k \in \mathbb{N}$ connect the root o_k of T_k to the vertex $k \in \mathbb{N}$ by an edge.

This constructs a graph, call it G.

It is simple to show (you should do this) that the leaves of T_k are at distance $k+1$ from $k \in \mathbb{N}$ and so at distance $2k+1$ from $0 \in \mathbb{N}$.

Thus, if $r \geq 2k+1$, all the trees $T_m, m \leq k$ and the vertices $\{0, 1, \ldots, k\}$ are in $B(0, r)$ (there are even more vertices in this ball, but we only require a lower bound). So for $r \geq 2k+1$,

$$|B(0,r)| \geq |T_0| + |T_1| + \cdots |T_k| + k + 1 = \sum_{m=0}^{k} (2^{m+1} - 1) + k + 1$$
$$= 2 \cdot \sum_{m=0}^{k} 2^m = 2 \cdot (2^{k+1} - 1).$$

This is greater than $\sqrt{2}^r$ for all $k \geq 2$.

So we are left with showing that $p_c(G) = 1$.

This is true for both site and bond percolation. Since $p_c^{\mathsf{bond}} \leq p_c^{\mathsf{site}}$ it suffices to prove it for bond. (However, in this case both are easy.)

If $p < 1$ and $\mathcal{C}$ is the component of 0 in p-bond percolation on G,

Note that since G is a tree, if $x \in T_k, y \in T_m$ for some $m \neq k$, then in bond percolation on G $x \leftrightarrow y$ if and only if $k \leftrightarrow m$. This is because any path from x to y must pass through both k and m.

For every $x \in G$ let $\rho(x)$ denote the unique $k \in \mathbb{N}$ such that either $x = k$ or $x \in T_k$. We have $0 \leftrightarrow x$ if and only if $0 \leftrightarrow \rho(x)$.

Since all T_k are finite, we have that $\mathcal{C}(0)$ is infinite if and only if there are infinitely many $k \in \mathbb{N}$ such that $0 \leftrightarrow x$ for some $x \in T_k \cup \{k\}$. But this is if and only if there are infinitely many $k \in \mathbb{N}$ such that $0 \leftrightarrow k$.

That is, we have show that if $\mathcal{C}(0)$ is infinite in p-bond percolation on G, then there is an infinite component in p-bond percolation on $\mathbb{N}$ as a subgraph of G. That is, $p_c(\mathbb{N}) \leq p_c(G)$. Since $p_c(\mathbb{N}) = 1$ we are done. $\square$

Exercise 4. *Let G be an infinite connected simple graph (no self loops or multiple edges) and $o \in G$ some vertex. Suppose that G is d-regular, $d \geq 2$.*

In this exercise, it will be useful to consider the following set: Let $C_{n,b}$ be the set of all connected subsets of G that contain o and have exactly n vertices and boundary size b; that is all connected subsets S such that $o \in S$, $|S| = n$ and for

$$\partial S = \{ y \notin S \;:\; y \sim S \}$$

$|\partial S| = b$.

(A) *Show that if $C_{n,b} \neq \emptyset$ then $b \leq dn$. Show that if $n = 1$ then $b = d$ and if $n \geq 2$ then $b \leq (d-1)n$.*

(B) *Show that for any $p \in (0,1)$,*

$$\sum_{n,b} |C_{n,b}| p^n (1-p)^b \leq 1.$$

(C) *Let A_n be the set of connected subsets of G that contain o and have exactly n vertices. Show that*

$$|A_n| \leq \left(\tfrac{d^d}{(d-1)^{d-1}} \right)^n .$$

Solution to Exercise 4.

(A) Let $S \in C_{n,e}$ then

$$dn = \sum_{x \in S} \sum_{y \sim x} 1 = \sum_{x \in S} \sum_{y \sim x} \mathbf{1}_{\{y \in S\}} + \mathbf{1}_{\{y \notin S\}} = |\{ x \sim y \;:\; x, y \in S \}| + b.$$

Any connected graph on n vertices has at least $n-1$ edges. $|\{ x \sim y \;:\; x, y \in S \}|$ is exactly twice the number of edges in S as a subgraph. So $dn \geq 2(n-1)+b$ and $b \leq (d-2)n + 2$.

When $n = 1$ the set is just $S = \{o\}$ and $b = d$. When $n > 1$ then $b \leq (d-2)n + 2 \leq (d-1)n$.

(B) Consider p-site percolation on G. Then,

$$1 \geq \mathbb{P}_p[|\mathcal{C}(o)| < \infty] = \sum_n \mathbb{P}_p[|\mathcal{C}(o)| = n] = \sum_n \sum_b \sum_{S \in C_{n,b}} p^n (1-p)^b$$

(C) If $n = 1$ then $|A_1| = 1$.

If $n > 1$: for any $p \in (0,1)$ we have that

$$|A_n| = \sum_b |C_{n,b}| \leq \sum_n |C_{n,b}| p^n (1-p)^b \cdot p^{-n}(1-p)^{-(d-1)n} \leq \big(p(1-p)^{(d-1)}\big)^{-n}.$$

Maximizing $p(1-p)^{(d-1)}$ gives $p = \frac{1}{d}$, so

$$|A_n| \leq \left(d\left(\tfrac{d}{d-1}\right)^{d-1}\right)^n.$$

□

Exercise 5. *Let $d > 1$. In this exercise we will show in steps that in site percolation on $\mathbb{Z}^d$, for $p_c = p_c(\mathbb{Z}^d)$, there exist a constant $c = c(d) > 0$ such that*

$$\mathbb{P}_{p_c}[0 \leftrightarrow \partial_r(0)] \geq c r^{(1-d)/2}.$$

That is, the probability to be connected to distance r does not decay exponentially (as in the sub-critical case).

(1) *We say that a collection $(\Omega(x))_{x \in \mathbb{Z}^d}$ is a (p,Δ)-***almost independent** *percolation if:*
- *For every x, $\Omega(x)$ is a Bernoulli random variable, with $\mathbb{E}[\Omega(x)] = \mathbb{P}[\Omega(x) = 1] \leq p$ (that is, the probability that x is open is at most p).*
- *For every two subsets $A, B \subset \mathbb{Z}^d$ such that $\operatorname{dist}(A,B) > \Delta$ we have that $(\Omega(x))_{x\in A}$ is independent of $(\Omega(x))_{x \in B}$. (Here the distance is the graph distance in $\mathbb{Z}^d$.)*

Note that in a (p,Δ)-almost independent percolation, it is not necessarily true that vertices close to one another are independent.

Show that if S is a finite connected subset of $\mathbb{Z}^d$ containing 0, then in a (p,Δ)-almost independent percolation Ω,

$$\mathbb{P}[S \text{ is open in } \Omega] \leq p^{|S|/V},$$

where $C(z,r) := \{z \;:\; ||z||_\infty \leq r\}$ and $V = |C(0,\Delta)|$ is the size of a L^∞-ball of radius Δ in $\mathbb{Z}^d$.

(2) *Show that for any* d, Δ *there exist* $p, c_1, c_2 > 0$ *such that if* Ω *is a* (p, Δ)*-almost independent percolation on* $\mathbb{Z}^d$*, then in* Ω*:*

$$\mathbb{P}[0 \leftrightarrow \partial_r(0) \text{ in } \Omega] \leq \mathbb{P}[|\mathcal{C}_\Omega(0)| \geq r] \leq c_1 e^{-c_2 r}.$$

(It may be useful to use part (C) of Exercise 4.)

(3) *Consider* p*-site percolation on* $\mathbb{Z}^d$ *(totally independent case). Use the previous items to show that there exist* $q, c_1, c_2 > 0$ *such that the following holds. If there exists* $r > 0$ *such that*

$$\mathbb{P}_p[C(0, r) \leftrightarrow \mathbb{Z}^d \setminus C(0, 3r)] \leq q$$

then for all $R > 0$*,*

$$\mathbb{P}_p[C(0, R) \leftrightarrow \mathbb{Z}^d \setminus C(0, 3R)] \leq c_1 e^{-c_2 R/r}.$$

(Hint: Define an appropriate almost independent percolation. Note that a tessellation of $\mathbb{Z}^d$ *by* L^∞*-balls of radius* r *has an isomorphic graph structure to that of* $\mathbb{Z}^d$*.)*

(4) *Consider* p*-site percolation on* $\mathbb{Z}^d$*. Use the BK inequality to show that there exist* $q, c_1, c_2 > 0$ *such that if there exists* $r > 0$ *such that*

$$\mathbb{P}_p[0 \leftrightarrow \partial_r(0)] \leq q r^{(1-d)/2}$$

then for all $R > 0$

$$\mathbb{P}_p[0 \leftrightarrow \partial_R(0)] \leq c_1 e^{-c_2 R/r}.$$

(5) *Conclude that for* $p = p_c$ *there exists* $q > 0$ *such that for all* $r > 0$*,*

$$\mathbb{P}_{p_c}[0 \leftrightarrow \partial_r(0)] \geq q r^{(1-d)/2}.$$

Solution to Exercise 5.

(1) Let $V := |C(0,\Delta)|$ be the size of a L^∞-ball of radius Δ in $\mathbb{Z}^d$. So $|V| \leq (2\Delta+1)^d$.

Suppose that $S \subset \mathbb{Z}^d$ is a finite connected subset of $\mathbb{Z}^d$. By repeatedly removing balls of radius Δ from S, one may find a subset $A \subset S$ such that $|A| \geq V^{-1} \cdot |S|$ and for any two vertices $a \neq b \in A$ we have that $\mathrm{dist}(a,b) > \Delta$.

If Ω is (p,Δ)-almost independent percolation, we have that $(\Omega(a))_{a\in A}$ are all independent Bernoulli random variables, with mean at most p. So

$$\mathbb{P}[S \text{ is open in } \Omega] \leq \mathbb{P}[A \text{ is open in } \Omega] \leq p^{|A|} \leq p^{|S|/V}.$$

(2) Let Σ_n be the set of all S such that S is a finite connected subset of $\mathbb{Z}^d$ with $|S| = n$. By part (C) of Exercise 4, $|\Sigma_n| \leq C^n$ for some $C > 0$.

Choose p small enough so that $Cp^{1/V} < 1$.

Now, for any $S \in \Sigma$, if Ω is (p,Δ)-almost independent percolation, then $\mathbb{P}[S \text{ is open in } \Omega] \leq p^{n/V}$. So

$$\mathbb{P}[|\mathcal{C}_\Omega(0)| \geq r] \leq \mathbb{P}[\exists\ n \geq r, S \in \Sigma_n \ :\ \mathcal{C}(0) = S] \leq \sum_{n\geq r} \sum_{S\in\Sigma_n} \mathbb{P}[S \text{ is open in } \Omega]$$
$$\leq \sum_{n\geq r} C^n p^{n/V} \leq (Cp^{1/V})^r \cdot \frac{1}{1 - Cp^{1/V}}.$$

Taking $c_1 = \frac{1}{1-Cp^{1/V}}$ and $c_2 = -\log(Cp^{1/V})$ completes this item.

(3) Let $B = C(0,r)$. Recall that this is the L^∞-cube, $B = \left\{x \in \mathbb{Z}^d \ :\ ||x||_\infty \leq r\right\}$. For every $z \in \mathbb{Z}^d$ let $B_z := B + 2rz$. Write $B_z \sim B_w$ if $z \sim w$ in $\mathbb{Z}^d$. Note that because $\mathrm{dist}(2rz, 2rw) = 2r\mathrm{dist}(z,w)$ we have that $B_z \cap B_w \neq \emptyset$ if and only if $B_z \sim B_w$ and the intersection is exactly at this points which are at distance exactly r from both z and w.

The balls $(B_z)_{z\in\mathbb{Z}^d}$ cover $\mathbb{Z}^d$ and form a graph isomorphic to $\mathbb{Z}^d$.

Consider p-site percolation on $\mathbb{Z}^d$. Define a configuration Ω on $\mathbb{Z}^d$ by setting $\Omega(z) = 1$ if and only if $B_z = C(2rz, r) \leftrightarrow \mathbb{Z}^d \setminus C(2rz, 3r)$ in the original site percolation. So $\mathbb{P}[\Omega(z) = 1] \leq \mathbb{P}[C(0,r) \leftrightarrow \mathbb{Z}^d \setminus C(0,3r)]$. Also, $\Omega(z)$ only depends on the state of the vertices in $B(2rz, 3r+1)$. So if $\text{dist}(z,w) > 4$ then $\text{dist}(2rz, 2rw) > 8r > 2(3r+1)$ which implies that $C(2rz, 3r+1) \cap C(2rw, 3r+1) = \emptyset$, and so $\Omega(z), \Omega(w)$ are independent.

Moreover, if $\text{dist}(A, A') > 4$ then taking $M = \bigcup_{a \in A} C(2ra, 3r+1)$ and $M' = \bigcup_{a' \in A'} C(2ra', 3r+1)$ we have that $(\Omega(a))_{a \in A} \in \mathcal{F}_M$ and $(\Omega(a'))_{a' \in A'} \in \mathcal{F}_{M'}$. Since $M \cap M' = \emptyset$, we have that $(\Omega(a))_{a \in A}$ is independent of $(\Omega(a))_{a \in A}$.

So we have that Ω is a (q, Δ)-almost independent percolation on $\mathbb{Z}^d$ with $q = \mathbb{P}[C(0,r) \leftrightarrow \mathbb{Z}^d \setminus C(0,3r)]$ and $\Delta = 4$.

Thus, if there exists r such that q is small enough, we have that for all $R > 0$,

$$\mathbb{P}[0 \leftrightarrow \partial_R \text{ in } \Omega] \leq c_1 e^{-c_2 R}.$$

Now back in the original independent p-site percolation, suppose that $C(0,R) \leftrightarrow \mathbb{Z}^d \setminus C(0,3R)$. This path must pass through order R/r annuli of the form $C(2rz, 3r+1)$, connecting in each one $C(2rz, r)$ to $\mathbb{Z}^d \setminus C(2rz, 3r)$. Indeed, let γ be an open path in $\mathbb{Z}^d$ connecting $B(0,R)$ to $\mathbb{Z}^d \setminus C(0,3R)$ in the original independent p-site percolation. Then, suppose that z is such that $\text{dist}(2rz, \gamma) \leq r$ and $\text{dist}(2rz, C(0,R)) > 3r$. Then $C(2rz, r) \leftrightarrow \mathbb{Z}^d \setminus C(2rz, 3r)$ so $\Omega(z) = 1$. Similarly, if $\text{dist}(2rz, \gamma) \leq r$ and $\text{dist}(2rz, \mathbb{Z}^d \setminus C(0,3R)) > 3r$ then $\Omega(z) = 1$. Thus, by following γ and noting each time it reaches a new ball $C(2rz, r)$ we have in Ω an open path of length at least $\frac{3R}{r} - 1$.

Thus,

$$\mathbb{P}_p[C(0,R) \leftrightarrow \mathbb{Z}^d \setminus C(0,3R)] \leq c_1 e^{-c_2 R/r},$$

perhaps by modifying appropriately c_1, c_2.

(4) Let $c > 0$ be such that $|\partial C(0,r)| \leq c r^{d-1}$ for all r. Suppose that for some r,

$$(\mathbb{P}_p[0 \leftrightarrow \partial_r(0)])^2 \leq q c^{-1} r^{1-d}.$$

If $C(0,r) \leftrightarrow \mathbb{Z}^d \setminus C(0,3r)$ then there is an open simple path going from $C(0,r)$ to outside $C(0,3r)$. Letting y be the first point on this path that is in $\partial C(0,2r)$ we have that y is connected to $C(0,r)$ and to $\mathbb{Z}^d \setminus C(0,3r)$ by two open disjoint paths. With the BK inequality we have that

$$\begin{aligned}\mathbb{P}_p[C(0,r) \leftrightarrow \mathbb{Z}^d \setminus C(0,3r)] &\leq \mathbb{P}_p[\exists\, y \in \partial C(0,2r) \ : \ \{y \leftrightarrow \partial_r(y)\} \circ \{y \leftrightarrow \partial_r(y)\}] \\ &\leq |\partial C(0,r)| \cdot (\mathbb{P}_p[0 \leftrightarrow \partial_r(0)])^2 \leq q.\end{aligned}$$

If q was small enough, this would imply that for all $R > 0$,

$$\mathbb{P}_p[C(0,R) \leftrightarrow \mathbb{Z}^d \setminus C(0,3R)] \leq c_1 e^{-c_2 R/r}.$$

But

$$\mathbb{P}_p[0 \leftrightarrow \partial_R(0)] \leq \mathbb{P}_p[C(0,R/3) \leftrightarrow \mathbb{Z}^d \setminus C(0,R)].$$

(5) The previous item tells us that if for some small enough q and some $r > 0$,

$$\mathbb{P}_p[0 \leftrightarrow \partial_r(0)] \leq q r^{(1-d)/2}$$

then $\mathbb{P}_p[0 \leftrightarrow \partial_R(0)]$ decays exponentially in R. Since balls in $\mathbb{Z}^d$ grow polynomially in the radius, this would imply that $\mathbb{E}_p[|\mathcal{C}(0)|] < \infty$. Since this cannot happen at $p = p_c$ we get that for some small enough q it must be that for any $r > 0$

$$\mathbb{P}_p[0 \leftrightarrow \partial_r(0)] \geq q r^{(1-d)/2}.$$

$\square$

Bonus Exercise

- This exercise is with 100 points solved. That means if you solve it, you get 100 and do not need to solve the other exercises.
- Better to show me a solution first, just in case...
- At the moment I do not know how to solve it, so it may (or may not, who knows?) be difficult.
- You are not allowed to solve it for high dimensions using the method known as "lace expansion".

Bonus Exercise. A random walk on $\mathbb{Z}^d$ is a sequence of vertices in $\mathbb{Z}^d$, say $(Z_n)_n$ such that for all n,

$$\mathbb{P}[Z_{n+1} = y \mid Z_n = x, Z_{n-1}, \ldots, Z_0] = \mathbb{P}[Z_{n+1} = y \mid Z_n = x] = \tfrac{1}{2d}\mathbf{1}_{\{y \sim x\}}.$$

For a set $S \subset \mathbb{Z}^d$ we define

$$T_S = \inf \{ n \geq 1 \ : \ Z_n \in S \} .$$

(Note that we don't count time $n = 0$.)

Given a subset $S \subset \mathbb{Z}^d$ we define the **index** of S to be the number

$$I(S) := \mathbf{E}[T_S \mid Z_0 = 0],$$

where $\mathbf{E}$ is expectation with respect to the random walk measure. ($I(S)$ may be infinite.)

Consider Ω_p, p-percolation on $\mathbb{Z}^d$. For every $p \in (0,1)$ define a random subset: If $\theta(p) > 0$ then let S_p be the (unique) infinite component in Ω_p. If $\theta(p) = 0$ let $S_p = \emptyset$.

Prove the following for some $d \geq 3$.

- Show that if $p > p_c(\mathbb{Z}^d)$ then $\mathbb{P}_p$-a.s. $I(S_p)$ is finite.
- Prove or provide a counter-example: If $\theta(p) > 0$ then $\mathbb{P}_p$-a.s. $I(S_p)$ is finite.
- Prove that $\mathbb{E}_{p_c}[I(S_{p_c})] = \infty$.