Percolation

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Exercise Sheet # 2

Exercise 1. *Show that the event that there exists an infinite component is translation invariant.*

Solution to Exercise 1. If $\varphi \in \mathrm{Aut}(G)$ then φ maps infinite connected subsets to infinite connected subsets. So if ω is a subgraph containing an infinite component, then $\varphi\omega$ also contains an infinite component. Also, if ω contains only finite components, then $\varphi\omega$ contains only finite components.

Let A be the event that there exists an infinite component. Then the above is just $\omega \in A \iff \varphi\omega \in A$, which implies $A = \varphi A$.

This holds for all $\varphi \in \mathrm{Aut}(G)$ so A is translation invariant. □

Exercise 2. *Let G be an infinite transitive graph, and let $E \subset E(G), |E| < \infty$ be some finite subset. Then, there exists $\varphi \in \mathrm{Aut}(G)$ such that $\varphi E \cap E = \emptyset$.*

Solution to Exercise 2. Fix some vertex $x \in G$. Let $r = \max\{\mathrm{dist}(e,x) \ : \ e \in E\}$. Let $R > 3r$ and choose a vertex $y \in G$ such that $\mathrm{dist}(x,y) > R$. Let $\varphi \in \mathrm{Aut}(G)$ be such that $\varphi(x) = y$.

Then, since φ is a graph automorphism, it preserves distances. So for any edge e such that $\mathrm{dist}(e,x) \leq r$, we have that $\mathrm{dist}(\varphi(e),y) \leq r$ and so $\mathrm{dist}(\varphi(e),x) > R - r > r$. Thus, for any $e \in E$ we have that $\varphi(e) \notin E$. That is, $\varphi E \cap E = \emptyset$. □

Exercise 3. *Show that $\{x \leftrightarrow \infty\}$ is an increasing event.*

Show that $\{x \leftrightarrow y\}$ is an increasing event.

Show that A is increasing if and only if A^c is decreasing.

Show that the union of increasing events is increasing.

Show that the intersection of increasing events is increasing.

Show that $\{x$ is an isolated vertex $\}$ is a decreasing event.

Give an example of an event that is not increasing or decreasing.

Solution to Exercise 3. If $\omega \leq \eta$ and ω is such that $\omega \in \{x \leftrightarrow \infty\}$, then the infinite component of x in ω is open in η, so η also contains an infinite component for x.

In general, if $\omega \leq \eta$, then for every z, the component of z in ω is contained in the component of z in η. So if $x \leftrightarrow y$ in ω then $x \leftrightarrow y$ in η.

Let A be an increasing event, and let B be a decreasing event. Let $\omega \leq \eta$. If $\eta \in A^c$, then $\eta \notin A$, so it cannot be that $\omega \in A$, which implies that $\omega \in A^c$. If $\omega \in B^c$ then $\omega \notin B$ so $\eta \notin B$ (because B is decreasing) and so $\eta \in B^c$. Since this is true for all $\omega \leq \eta$, we get that A^c is decreasing and B^c is increasing.

Suppose that $(A_n)_n$ are increasing events. Let $A = \bigcup_n A_n$. Suppose that $\omega \in A$, and that $\eta \geq \omega$. Then, there exists n such that $\omega \in A_n$, and since A_n is increasing, also $\eta \in A_n$. So $\eta \in A$. Thus, A is increasing.

Let $B = \bigcap_n A_n$. If $\eta \geq \omega$ and $\omega \in B$ then $\omega \in A_n$ for all n. Since A_n are all increasing, $\eta \in A_n$ for all n. So $\eta \in B$.

The event that x is an isolated vertex is the event that $x \not\leftrightarrow y$ for all $y \sim x$. So the intersection of decreasing events. That is, the event that x is an isolated vertex is the complement of the union of increasing events, and so a decreasing event.

Consider the event $A = \{x \leftrightarrow \infty, \deg(x) = 1\}$. Then opening edges adjacent to x ruins the event, however, closing edges may disconnect x from infinity, so A is neither increasing nor decreasing. □

Exercise 4. *Let G be a graph. A function $f : \{0,1\}^{E(G)} \to \mathbb{R}$ is* **increasing** *if $\omega \leq \eta$ implies $f(\omega) \leq f(\eta)$.*

Show that for an event A, $\mathbf{1}_A$ is increasing if and only if A is an increasing event.

Solution to Exercise 4. Let $f = \mathbf{1}_A$.

Assume that A is increasing. For any $\omega \leq \eta$, if $\omega \notin A$ then $f(\omega) = 0 \leq f(\eta)$. If $\omega \in A$ then since A is increasing $\eta \in A$ and so $f(\omega) = 1 = f(\eta)$. Since this holds for all $\omega \leq \eta$, we get that f is increasing.

Now assume that f is increasing. Let $\omega \leq \eta$, and assume that $\omega \in A$. So $1 = f(\omega) \leq f(\eta)$ which implies that $f(\eta) = 1$ and so $\eta \in A$. Since this holds for all $\omega \leq \eta$, we get that A is increasing. □

Exercise 5. *Show that* $p_c(\mathbb{Z}) = 1$.

Solution to Exercise 5. Let $p < 1$. It suffices to show that $\Theta_{\mathbb{Z}}(p) = 0$.

First we investigate the event $\{0 \leftrightarrow \infty\}$. Let A_n be the event that both edges $\{n, n+1\}$ and $\{-n, -(n+1)\}$ are closed. So $\mathbb{P}_p[A_n] = (1-p)^2$. Since for different n these edges are different, we have that $(A_n)_n$ are independent, and also $(A_n^c)_n$ are independent. Thus,

$$\mathbb{P}_p[\bigcap_n A_n^c] = \lim_{N \to \infty} \mathbb{P}[\bigcap_{n=1}^N A_n^c] = \lim_{N \to \infty} \prod_{n=1}^N [1 - (1-p)^2] = 0$$

because for $p < 1$ we have $1 - (1-p)^2 < 1$. Thus,

$$\mathbb{P}_p[\exists n \ : \ A_n] = 1.$$

That is, $\mathbb{P}_p$-a.s. there exists n such that both $\{n, n+1\}$ and $\{-n, -(n+1)\}$ are closed. This implies that $\mathbb{P}_p$-a.s. $\mathcal{C}(0) \subset [-n, n]$ and so finite. Thus, $\mathbb{P}_p[0 \leftrightarrow \infty] = 0$.

Now, there was nothing special about the vertex 0 in this argument. One could replace 0 with any other vertex. So, we conclude that for any $x \in \mathbb{Z}$, $\mathbb{P}_p[x \leftrightarrow \infty] = 0$. Summing over all x we have,

$$\Theta_{\mathbb{Z}}(p) = \mathbb{P}_p[\exists \ x \ : \ x \leftrightarrow \infty] \leq \sum_x \mathbb{P}_p[x \leftrightarrow \infty] = 0.$$

□