Percolation

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Exercise Sheet # 7

Exercise 1. *Show that any group acts on itself by left multiplication.*

Show that any group acts on itself by conjugation.

Solution to Exercise 1. The first action is just associativity of the group multiplication.

The second action follows since for any $g, h, x \in \Gamma$,

$$gh.x = h^{-1}g^{-1}xgh = h.(g.x),$$

and $1_\Gamma.x = 1_\Gamma^{-1}x1_\Gamma = x$. $\square$

Exercise 2. *Let* $\Gamma = \mathrm{Aut}(G)$ *be the set of automorphisms of a graph* G. *Show that* Γ *is a group that acts on* G.

Solution to Exercise 2. The action is just $\varphi.x = \varphi(x)$. This is associative because composition of automorphisms is associative. The unit element in the group is the identity which acts trivially. $\square$

Exercise 3. *Show that the stabilizer of* x *is a subgroup.*

Show that $|\Gamma x| = [\Gamma : \Gamma_x]$.

Show that if $y = g.x$ *then* $G_y = gG_xg^{-1}$.

Solution to Exercise 3. If $g, h \in \Gamma_x$ then $x = h^{-1}hx = h^{-1}x$ so $h^{-1} \in \Gamma_x$ and also $gh^{-1}x = gx = x$ so $gh^{-1} \in \Gamma_x$. This shows that Γ_x is a subgroup.

To show $|\Gamma x| = [\Gamma : \Gamma_x]$ we need to show a bijection between cosets of Γ_x and elements in the orbit Γx.

For $g\Gamma_x$ some coset, set $f(g) := g.x \in \Gamma x$. This is injective, since if $g.x = h.x$ then $h^{-1}g \in \Gamma_x$ so $g \in h\Gamma_x$ which implies that $g\Gamma_x = h\Gamma_x$ are the same coset. Also this map is onto, because for any $g.x \in \Gamma x$ we have $g\Gamma_x \mapsto g.x$.

For the last assertion, let $y = g.x$. So $x = g^{-1}.y$ by associativity. Thus, by the symmetry between x and y, it suffices to show that $gG_xg^{-1} \subset G_y$. Indeed, if we consider ghg^{-1} for some $h \in G_x$ then $ghg^{-1}.y = gh.x = g.x = y$, so $ghg^{-1} \in G_y$. $\square$

Exercise 4. *Let Γ be a group acting on X. Show that for all $x, y \in X$,*

$$|\Gamma_x y| = [\Gamma_x : \Gamma_x \cap \Gamma_y].$$

A discrete action is one where all stabilizer subgroups are finite; i.e. $|\Gamma_x| < \infty$ *for all $x \in X$. Show that if Γ is a discrete action on X then Γ is unimodular.*

Solution to Exercise 4. The group Γ_x acts on X. $\Gamma_x \cap \Gamma_y$ is the stabilizer subgroup of y under this action. So we have already seen that $[\Gamma_x : \Gamma_x \cap \Gamma_y] = |\Gamma_x y|$.

If Γ is a discrete action, then we have for $G = \Gamma_x \cap \Gamma_y$ (using Langrage's Theorem),

$$\frac{|\Gamma_x y|}{|\Gamma_y x|} = \frac{[\Gamma_x : G]}{[\Gamma_y : G]} = \frac{|\Gamma_x|}{|\Gamma_y|}.$$

Now, if $y \in \Gamma x$ then there exists $g \in \Gamma$ such that $y = g.x$. So $\Gamma_y = g\Gamma_x g^{-1}$ and so $|\Gamma_x| = |\Gamma_y|$. $\square$

Exercise 5. *Let G be a Cayley graph. Show that both site and bond percolation on G are invariant percolation.*

Solution to Exercise 5. We prove for bond percolation. Site is similar.

If $\varphi \in \mathrm{Aut}(G)$ and Ω_p is p-bond percolation on G, then $(\varphi\Omega_p(e) = \Omega_p(\varphi^{-1}(e)))_{e \in E(G)}$ are independent Bernoulli-p random variables, so $\varphi\Omega_p$ is also a Bernoulli bond percolation. $\square$

Exercise 6. *Show that for the d-regular tree $p_u(\mathbb{T}_d) = 1$.*

Solution to Exercise 6. Let $T^1, \ldots, T^d$ be the subtrees rooted at the children of the root $o \in \mathbb{T}_d$. We have already seen that $p_c(T^j) = p_c(\mathbb{T}_d) = \frac{1}{d-1}$. So for any $p > p_c$, each subtree T^j contains an infinite component a.s.

By forcing the root to be closed (or for bond percolation forcing the edges between the root and its children to be closed) we get that with positive probability there are many infinite components, which implies that there must be infinitely many of them. $\square$