Some Remarks on the co-volume of Lattices Acting on a Product of Trees.

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1 Introduction

Many of the works concerning the automorphism group of a regular tree $\mathrm{Aut}(T^n)$ and its uniform lattices suggest a strong resemblance between $\mathrm{Aut}(T^n)$ and simple rank one real Lie groups, and even more so between $\mathrm{Aut}(T^n)$ and the p-adic points of a rank one algebraic group G. Under this analogy the automorphism group of a product of two trees $A = \mathrm{Aut}(T^n \times T^m) = \mathrm{Aut}(\Delta)$ corresponds to a product of two simple rank one Lie groups. Lately an elaborate theory of "irreducible uniform lattices" in A has been developed by Burger, Mozes and Zimmer [BM00, BMZ]. In this theory one can find analogs to many theorems pertaining to Lie groups such as Margulis' Super rigidity and arithmeticity Theorems.

Given a connected semisimple Lie group without compact factors - G Kazhdan and Margulis (see [KM68] and [Rag72]) have shown that, there is a lower bound on the co-volume of lattices in G. Explicit calculations of such bounds were made in many cases (see [Sie45, Lub90]). An analogous theorem does not hold, however, for the automorphism group of a regular tree. In fact examples for families of uniform lattices $\Gamma^i \lneqq \Gamma^{i+1} \ldots$ in the automorphism group of any regular tree can be found in [BK90].

The main idea of the work of Kazhdan and Margulis in the case of uniform lattices is the following. Given a connected semi simple Lie group without compact factors G, we can find a so called, Zesenhaus neighborhood of the identity $1 \in U \subset G$. Such that for every discreet subgroup $\Gamma < G$ the elements of $\Gamma \cap U$ are contained in a connected nilpotent Lie subgroup of G say N. All elements of the connected Nilpotent group are unipotent, By Engel's theorem. If Γ is a uniform lattice then it contains no unipotent elements (except perhaps for a cyclic group of a bounded order) therefore $U \cap \Gamma$ has a bounded order.

If we restrict our attention to a subset of all uniform lattices in $\mathrm{Aut}(T)$ we might obtain minimal co-volume theorems.

Take, for example, the following situation (see [Lub90]). Let F be a non archemedian local field with residue field of order $q = p^{\alpha}$, O its ring of integers. Let $G = SL_2(F) < \mathrm{Aut}(T^{q+1})$ and consider only uniform lattices $\Gamma < G$. Pick a vertex $v \in T$ then $G_v = SL_2(\mathcal{O})$, the stabilizer of a point, is virtually pro-p. In fact the kernel $U = Ker\{SL_2(\mathcal{O}) \to SL_2(\mathbb{F}_q)\}$ is a pro-p group which is a compact open neighborhood of the identity. Every finite subgroup of U must be a p group. All p-groups in G consist of unipotent elements and a uniform lattice contains no such elements so $\Gamma \cap U = 1$ for any uniform lattice $\Gamma < G$.

In the spirit of these ideas it is worthwhile to formulate the Thompson Wielandt theorem in similar terms. Here again we have a family of lattices that intersect a neighborhood of the identity in a p-group.

Definition 1.1. *A subgroup $\Gamma < \mathrm{Aut}(T)$ is called locally primitive if the action of every vertex stabilizer on the link of the corresponding vertex is primitive.*

Definition 1.2. *A lattice $\Gamma < \mathrm{Aut}(T^1) \times \mathrm{Aut}(T^2)$ is called irreducible if its projections on both factors are non discreet.*

Definition 1.3. *A lattice $\Gamma < \mathrm{Aut}(T^1) \times \mathrm{Aut}(T^2)$ is called locally primitive if its projections are locally primitive in the sense of definition 1.1.*

Theorem 1.4. *Let $T = T^n$ be the n-regular tree, $e \in E(T)$ be an edge, $B_T(e, n)$ the set of all vertices of T at a distance $\leq n$ from e. Let $U < \mathrm{Aut}(T)$ be the pointwise stabilizer of $B_T(e, 2)$, which is an open compact neighborhood of the identity. Then for every locally primitive lattice $\Gamma < \mathrm{Aut}(T)$ the group $\Gamma \cap U$ is either empty or an l-group for some prime $l < n$.*

For a proof see [Fan86b] and [BCN80]. Unfortunately the ideas of Kazhdan-Margulis have no immediate application here. We do not know anything about the value of l and even if we did, it is not true that a nilpotent subgroup or an l-subgroup of $\mathrm{Aut}(T)$ consists of unipotent elements. The similarity of the two fenomena, however, may lead to the conjecture that there exists a lower bound on the co-volume of locally primitive lattices in $\mathrm{Aut}(T)$.

The conjecture that there exists a minimal bound for co-volume of locally primitive lattices in $\mathrm{Aut}(T)$ is better known as the Goldshmidt Sims Conjecture. This conjecture can be formulated also in the language of Bass-Serre theory: Given an edge indexed graph consisting of just one edge $\overset{n}{\longleftrightarrow}\!\!\overset{n}{}$, there exist only a finite number of finite effective groupings $G \hookleftarrow A \hookrightarrow H$ such that the action of $G *_a H$ on T^n is locally primitive. This conjecture has been proved by Goldshmidt in the case $n = 3$ [Gol80, DGS85].

In section 5 we formulate and prove a two dimensional analog of the Goldshmidt Sims conjecture. Explicitly we show that given a Square S there is only a finite number of finite effective groupings $G(S)$ of S such that the universal cover $\widetilde{G(S)}$ is a product of two regular trees of given prime valence, and such that $\Gamma = \pi_1(G(S), \sigma_0)$, the fundamental group, is a locally primitive irreducible lattice in the sense of definitions 1.2 and 1.3. This enables us to give a lower bound on the co-volume of irreducible locally primitive lattices admiting a square as a fundamental domain.

In the case of a single tree local primitivity as in definition 1.1 implies that the fundamental domain for the lattice action is an edge (assuming the lattice acts on the tree without inversion). In the case of a product of two trees we assume explicitly that the lattice admits a square as a fundamental domain. In both cases we restrict the family of lattices that we are dealing with by requiring that the local action (i.e. the action of every vertex stabilizer on the link of the corresponding vertex) will be transitive. This seems like a very artificial requirement.

The requirement for a lattice acting on a product of two trees to be irreducible and locally primitive is very natural, however. It is well known that if $\Gamma < G$ is a uniform lattice in a semi simple lie group, then it's projection on any simple factor of G is either discreet (in which case Γ is called reducible) or dense (in which case Γ is called irreducible). If Γ is a uniform lattice in $\mathrm{Aut}(\Delta)$ then its projections on the two factors are never dense. In this setting irreduciblity is defined in 1.2. But this turns out to be too weak an assumption for most purposes. Stronger notions of "irreducibility" are defined by imposing some transitivity requirements on the action of the projections, thus requiring the projections to be "big". Some possible definitions require the projections to act locally primitively, locally 2-transitively or locally infinitely transitively. All of these cases are investigated in the works of Burger and Mozes [BM00] [BMZ]

The purpose of all the work done in sections 2-4 is to reduce the assumptions on the local action of Γ on Δ. Unfortunately I did not succeed in getting rid of such local assumptions on the action all together.

We make a distinction between two types of lattices. torsion free lattices acting freely on Δ and lattices with torsion. In the first case $X \overset{\mathrm{def}}{=} \Gamma\backslash\Delta$ is a complicated complex whose fundamental group is Γ. In this case it is trivial that Γ has a co-volume bounded from below by 1.

In the second case there exist a short exact sequence:

$$1 \to N \to \Gamma = \pi_1(\Gamma(X), \sigma_0) \to \pi_1(X, \sigma_0) \to 1 \tag{1.1}$$

Where N is the subgroup of Γ generated by all vertex stabilizers. The Margulis' normal subgroup theorem suggests that whenever N is not trivial it is very large and hence the structure of X relatively simple. If we assume that the projections of Γ act locally infinitely transitively we can use Margulis' work to conclude that $\pi_1(X, \sigma_0)$ has Kazhdan's property (T). Using this fact it is easy to see (theorem 4.2) that if X is covered by a product of two trees (a fact that can be expressed as a restriction on the local action of Γ on the links of the vertices). then X is actually contractible.

Sections 2-4 of this paper show that if $\Gamma < \mathrm{Aut}(\Delta)$ is an irreducible locally primitive lattice and X is covered by a product of two trees. Then one of the following holds:

- $\Gamma = \pi_1(X, \sigma_0)$.
- X is a square.

This gives a minimal co-volume theorem for a certain subset of all locally primitive irreducible lattices acting on a product of two regular trees of prime valence. The minimal volume theorem is stated and proved in section 6. In Sections 2, 3 We develop the technical Bass-Serre theoretic background used to prove the dichotomy theorem mentioned above in section 4.

I believe that it is true that there exist a lower bound on the co-volume of locally primitive irreducible lattices acting on a product of two regular trees of prime valence. In section 7 a counter example is given if we do not assume locally primitive but only irreducible lattices acting on a product of two regular (though not prime) trees.

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2 Some Bass Serre Theoretic Lemmas.

In [Ser80] and [Bas93] the theory of graphs of groups is developed. In [Hae91] this theory is generalized to higher dimensional simplicial complexes. The theory presented in [Hae91] generalizes without any difficulty to the more general concept of polyhedral complexes of groups[1]. In particular the theory is suitable for the treatment of group actions without inversions on the product of two trees, encoding such an action in the form of a square complex of groups. I will assume the readers are familiar with [Hae91, Ser80, Bas93]. For the convenience of the reader who is not familiar with the Bass Serre theory I include an appendix, (A), with some of the most important definitions and theorems that will be used in this paper. A reffrence to the appendix as well as to Haefliger's paper [Hae91] will be given every time a new concept is introduced.

In this section I will collect some lemmas about group actions on products of trees that will be needed later on. All the notation and terminology pertaining to the bass Serre theory will be taken from [Hae91] and appendix A. We will assume throughout that all complexes are connected.

First we give a simple characterization of a "local nature" of square complexes of groups that are obtained by an action of a group on a product of two trees.

Definition 2.1. *We say that a square complex of groups (see section A.2)* satisfies the link condition *if for every vertex v the link $Lk(\tilde{v})$ (see section A.11) is a full bipartite graph $K_{p,q}$*

Theorem 2.2. *A square complex of groups $G(X)$ is developable (see section A.5) with a universal cover (see section A.10) which is a product of two trees if and only if it satisfies the link condition.*

Proof. The "only if" part is obvious. For the "if" observe that a complex satisfying the link condition is non positively curved and therefore developable see [Hae91, Gro87, Bri90]. The universal cover $Y \stackrel{\text{def}}{=} (\widetilde{G(x), \sigma_0})$ is a simply connected square complex all of whose links are full bi-partite graphs. It is well known that such a complex must be a product of two trees. □

The following proposition enables us to express a fundamental group of a graph of groups (see section A.9) as a semi-direct product of the fundamental group of the graph (forgetting the groups) with the (normal) subgroup generated by all stabilizers.

Proposition 2.3. *Let $G(X)$ Be a complex of groups. then there exists a short exact sequence.*

$$1 \longrightarrow N \longrightarrow \pi_1(G(X), \sigma_0) \xrightarrow{\xi} \pi_1(X, \sigma_0) \longrightarrow 1 \tag{2.1}$$

[1] In fact an even farther reaching generalization is suggested there in terms of small categories without loops and their geometrical realizations.

where $N \triangleleft \pi_1(G(X), \sigma_0)$ is the subgroup generated by all vertex stabilizers. Furthermore in case X is 1-dimensional (i.e. a graph) this sequence naturally splits.

$$1 \longrightarrow \pi_1(X, \sigma_0) \longrightarrow \pi_1(G(X), \sigma_0) \tag{2.2}$$

Proof. **The existence of a short exact sequence is standard so I will not repeat the argument. The homomorphism $\xi : \pi_1(G(X), \sigma_0) \longrightarrow \pi_1(X, \sigma_0)$ takes the element $g_0e_0g_1 \dots g_n$ to $e_0, e_1, \dots, e_n$ (where e_i are edges in the barycentric subdivision of X, $i(e_0) = t(e_n) = \sigma_0$, $i(e_{i+1}) = t(e_i)$ and $g_i \in G_{i(e_i)}$). If X is a graph we obtain an immersion of graphs of groups (see section A.3 and definition A.6) $\Phi = (f, \phi_\eta, g_a) : X \hookrightarrow G(X)$ by taking f to be the identity map and ϕ_η, g_a to be trivial. Now By theorem A.5 we obtain an immersion of groups: $\delta\Phi_{\sigma_0} : \pi_1(X, \sigma_0) \hookrightarrow \pi_1(G(X), \sigma_0)$ and a $\delta\Phi_{\sigma_0}$ equivariant immersion of trees: $\widetilde{\delta\Phi} : (\widetilde{X, \sigma_0}) \hookrightarrow (\widetilde{G(X), \sigma_0})$. Using the definition of ξ and the definition of $\delta\Phi_{\sigma_0}$ in equation A.31 we see that $id_{\pi_1(X,\sigma_0)} = \xi \circ \delta\Phi_{\sigma_0}$ so $\delta\Phi_{\sigma_0}$ is the desired section over ξ.** □

remark It should be carefully noted that the second part of the theorem does not hold when the complex X is of higher dimension. Even an existence of an immersion $X \hookrightarrow G(X)$ is in general false. An easy example is the following. Take as a square complex the plane tessellated by squares. Choose one square there and look at the group of isometries of the plane generated by the four elements of order 2 rotating the plane $180°$ around each of the four vertices. The quotient of this action is a "sphere of groups" $G(X)$ but there is definitely no immersion $X \hookrightarrow G(X)$ because this would, by theorem A.5, give rise to a square complex morphism of the sphere X into the plain .

The following theorem gives the relationship between semi-direct product decomposition of proposition 2.3 and the homomorphism of groups induced by a morphism of graphs of groups given in theorem A.5.

Proposition 2.4. *Let $G(X) = (X, G_\sigma, \psi_a)$ and $G'(X') = (X', G'_\sigma, \psi'_a)$ be graphs of groups. $\Phi = (f, \phi_\sigma, g'_a) : G(X) \to G'(X')$ a morphism (see section A.3). Let $\pi_1(G(X), \sigma_0) = F \ltimes N$ and $\pi_1(G'(X'), f(\sigma_0)) = F' \ltimes N'$ be the semi direct product decompositions of proposition 2.3; and $\delta\Phi_{\sigma_0} : F \ltimes N \to F' \ltimes N'$ the homomorphism given by theorem A.5. Then the following hold:*

- $\delta\Phi_{\sigma_0}(N) \subset N'$
- *By the previous observation there is an induced homomorphism of groups $\overline{\delta\Phi_{\sigma_0}} : F \cong \pi_1(G(X), \sigma_0)/N \to F' \cong \pi_1(G'(X'), \sigma_0)/N'$. There is another*

natural morphism $f_* : F = \pi_1(X, \sigma_0) \to F' = \pi_1(X', f(\sigma_0)$ *induced on the fundamental groups by the graph morphism* f*. We claim that these morphisms are the same:*

$$f_* = \overline{\delta\Phi_{\sigma_0}} \tag{2.3}$$

Proof. $N \triangleleft \pi_1(G(X), \sigma_0)$ is generated as a normal subgroup by $\{G_\sigma | \sigma \in V(X)\}$. By the definition of $\delta\Phi_{\sigma_0}$ in equation A.31 $\delta\Phi_{\sigma_0}(G_\sigma) < G'_{f(\sigma)} < \{G'_\eta | \eta \in V(X')\} = N'$. This shows the first claim. It is now possible to define the quotient morphism in the second part of the claim. $(e_0, e_1, \ldots, e_n)$ is a typical element of F corresponding to $(e_0, e_1, \ldots, e_n)N$ under the isomorphism $F \cong \pi_1(G(X), \sigma_0)/N$. Now, using equation A.31 the later element is mapped by $\overline{\delta\Phi_{\sigma_0}}$ onto $(f(e_0), (g'_{e_0})^{-1}, f(e_1), \ldots, (g'_{e_n})^{-1})N' = (f(e_0), f(e_1), \ldots, f(e_n))N' \in \pi_1(G'(X'), \sigma_0)/N'$ the last equality is by normality $N' \triangleleft \pi_1(G'(X'), \sigma_0)$. On the other hand, $f_*(e_0, e_1, \ldots, e_n) = (f(e_0), f(e_1), \ldots, f(e_n))$. And a comparison of the two expressions completes the proof. $\square$

The proof of the following lemma uses the isomorphism of the two definitions of the fundamental group of a complex of groups (see theorem A.3, [Hae91, Section 3.1] and [Ser80]). Which gives a very short proof.

Lemma 2.5. *Let* $G(X)$ *be a graph of groups.* $S = \widetilde{G(X)}$ *its universal cover.* $x, y \in V(X)$ *vertices of the barycentric subdivision. We can identify* $\Lambda \cong \pi_1(G(X), y) \cong \pi_1(G(X), x) < \mathrm{Aut}(S)$ *as a subgroup of the automorphisms of* S*. For every* $\alpha \in \pi_1(G(X), x, y)$ *(see section A.9) there corresponds an isomorphism* $\mathrm{Ad}(\alpha) : \pi_1(G(X), y) \to \pi_1(G(X), x)$ *which induces an automorphism* $X(\alpha) : \Lambda \to \Lambda$ *we claim that* $X(\alpha)$ *is inner, i.e. there exists* $\chi(\alpha) \in Lambda$ *such that* $X(\alpha) = \mathrm{Ad}(\chi(\alpha))$*.*

Proof. Let $T \subset X$ be a maximal spanning tree. Construct the universal cover $S = (\widetilde{G(X), T}) = (\widetilde{G(X)})$ as in section A.10 (or [Hae91, Section 4]). Now it is natural to look at the fundamental group in the form $\pi_1(G(X), T)$ which is defined as a quotient of the path group $FG(X)$ (see sections A.7 and (A.9)). Look at the following diagram:

$$\begin{tikzcd}
\pi_1(G(X), y) \arrow[dd, "ad(\alpha)"'] \arrow[dr] & & \\
& FG(X) \arrow[r] & \pi_1(G(X), T) \\
\pi_1(G(X), x) \arrow[ur] & &
\end{tikzcd}$$

And take $\chi(\alpha)$ to be the image of $\alpha \in FG(X)$ in $\pi_1(G(X), T)$. This obviously gives the desired internal morphism. $\square$

3 Projection Maps

From now on we deal with the following situation.

- $\Delta = T^1 \times T^2$ is a product of two trees.
- $\mathrm{Aut}(\Delta) = \mathrm{Aut}(T^1) \times \mathrm{Aut}(T^2)$ its automorphism group [2].
- $\Gamma < \mathrm{Aut}(\Delta)$ a subgroup.
- $\phi^i : \mathrm{Aut}(\Delta) \to \mathrm{Aut}(T^i)$ the projections.
- $p^i : \Delta \to T^i$ the ϕ^i-equivariant projections.
- $\Gamma^i = \phi^i(\Gamma)$.
- $H^i = \overline{\Gamma^i}$ - the closure.
- $\Gamma(X) = (X, \Gamma_\sigma, \psi_a, \gamma_{a,b}) = \Gamma \backslash\backslash \Delta$ the quotient complex of groups formed as in section A.5.
- $\Gamma^i(X^i) = (X^i, H^i_\sigma, \xi^i_a) = \Gamma^i \backslash\backslash T^i$ the quotient graph of groups formed as in section A.5.

p^i,being ϕ^i invariant, induces a unique map f^i making the following diagram commutative.

$$\begin{tikzcd} \Gamma \arrow[r, "\phi^i"] \arrow[d] & \Gamma^i \arrow[d] \\ \Delta \arrow[r, "p^i"] \arrow[d] & T^i \arrow[d] \\ X \arrow[r, dashed, "\exists! f^i"] & X^i \end{tikzcd}$$

Our goal is to define a "morphism of complexes of groups" $\Gamma(X) \to \Gamma^i(X^i)$ over f^i encoding the morphism of actions $(\Gamma, \Delta) \to (\Gamma^i, T^i)$. We define a

[2] In the case where $T^1 \cong T^2$ the group $\mathrm{Aut}(T^1) \times \mathrm{Aut}(T^2)$ is in fact of index 2 in $\mathrm{Aut}(\Delta)$. In this case we will ignore the automorphism interchanging the two trees and denote by $\mathrm{Aut}(\Delta)$ the smaller group.

second graph of group structure $\widehat{\Gamma^i(X^i)}$ on X^i, induced by the map f^i. To every vertex $\sigma \in V(X^i)$ we associate the fundamental group of the graph of groups $(f^i)^{-1}(\sigma)$ and to every edge $a \in E(X^i)$ we associate a morphism of groups associated with a natural covering of the Graphs of groups corresponding to its initial and terminal vertices. We will show that the effective quotients of two graph of group structures on X^i are isomorphic.

Lemma 3.1. *$(f^i)^{-1}(\sigma)$ is connected for every vertex $\sigma \in V(X^i)$ of the barycentric subdivision of X^i*

Proof. Assume $f^i(\eta_1) = f^i(\eta_2) = \sigma$ let $\tilde{\eta_1}, \tilde{\eta_2}$ be liftings of η_1, η_2. Obviously $\{p^i(\tilde{\eta_j})\}_{j\in\{1,2\}}$ are both mapped to $\sigma (\mathrm{mod}\, \Gamma^i)$ so that there exist an element $\gamma_i \in \Gamma^i$ such that $\gamma_i p^i(\tilde{\eta_1}) = p^i(\tilde{\eta_2})$. Take γ satisfying $\gamma_i = \phi^i(\gamma)$ so that $p^i(\gamma\tilde{\eta_1}) = p^i(\tilde{\eta_2}) = p^i([\gamma\tilde{\eta_1}, \tilde{\eta_2}])$. The line $[\gamma\tilde{\eta_1}, \tilde{\eta_2}]$ is mapped $(\mathrm{mod}\, \Gamma)$ to a path connecting η_1 to η_2 inside $(f^i)^{-1}(\sigma)$. □

Definition 3.2. *Now we can define:*

1. *Let $\tau^i_* : V(X^i) \to V(X)$ be any section over f^i (taking σ to τ^i_σ).*
2. *Let $W^i_\sigma \stackrel{\text{def}}{=} (f^i)^{-1}(\sigma)$. W^i_σ is a connected sub-graph of X by lemma 3.1.*
3. *There is a natural structure of a graph of groups on W^i_σ induced from $\Gamma(X)$. Let us denote it by $\widehat{H^i_\sigma}(W^i_\sigma) = (W^i_\sigma, \Gamma_\eta, \psi_a, \gamma_{a,b})$.*
4. *Define $\widehat{H^i_\sigma} \stackrel{\text{def}}{=} \pi_1(\widehat{H^i_\sigma}(W^i_\sigma), \tau^i_\sigma)$*

The following proposition will be the main tool for defining the injective morphisms of the new graph of groups.

Proposition 3.3. *Let a be an edge in the barycentric subdivision of X^i. Let $i(a) = \epsilon; t(a) = \sigma$. There is a natural graph morphism $l^{i,a} : W^i_\epsilon \to W^i_\sigma$ and a natural covering-morphism (see definition A.6) of graphs of groups defined over $l^{i,a}$ denoted $\widehat{\Xi^i_a} = (l^{i,a}, \widehat{\phi^{i,a}_\eta}, \widehat{\delta^{i,a}_b}) : \widehat{H^i_\epsilon}(W^i_\epsilon) \to \widehat{H^i_\sigma}(W^i_\sigma)$*

Lemma 3.4. *Look at $(f^i)^{-1}(a)$ then the boundary is given by: $\partial(f^i)^{-1}(a) = (f^i)^{-1}(\epsilon) \cup (f^i)^{-1}(\sigma)$ and for every $x \in (f^i)^{-1}(\epsilon)$ there exists a unique point denoted by $l^{i,a}(x) \in (f^i)^{-1}(\sigma)$ such that $f^{3-i}(x) = f^{3-1}(l^{i,a}(x))$*

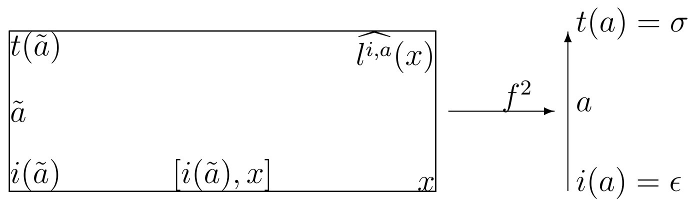

Figure 1: A strip in $(f^i)^{-1}(a)$

Proof of the lemma. **Let $\tilde{a}$ be any lifting of a to X. (i.e. $f^i(\tilde{a}) = a$ and $f^{3-i}(\tilde{a}) = \{pt\}$) an existence of such an $\tilde{a}$ is obvious by passing to the universal cover. There is a unique strip in $(f^i)^{-1}(a)$ bounded by $\tilde{a}$ on the one side and by $[i(\tilde{a}), x]$ on the other side, where $[i(\tilde{a}), x] \subset W^i_\epsilon$ is a geodesic in W^i_ϵ, it exists because by lemma 3.1 W^i_ϵ is connected. define $l^{i,a}(x)$ to be the fourth vertex of this strip as shown in figure 1. This proves existence. For uniqueness look at the graph $(f^{3-1})^{-1}(f^{3-i}(x))$ and assume both edges of its barycentric subdivision originating in x are mapped by f^i onto a, which is absurd.** $\square$

Proof of the proposition Lemma 3.4 gives the definition of $l^{i,a}$. Define $\widehat{\phi^{i,a}_\eta}$ for $\eta \in V(W^i_\epsilon)$ by

$$\widehat{\phi^{i,a}_\eta} \stackrel{\text{def}}{=} \psi_{[\eta, l^{i,a}_\eta]} \tag{3.1}$$

define $\widehat{\delta^{i,a}_b}$ by

$$\begin{aligned} \widehat{\delta^{i,a}_b} &\stackrel{\text{def}}{=} \gamma_{b,[t(b),l^{i,a}(t(b))]} \left(\gamma_{[i(b),l^{i,a}(i(b))],l^{i,a}(b)}\right)^{-1} \\ &= \gamma_{b,e}\left(\gamma_{c,d}\right)^{-1} \end{aligned} \tag{3.2}$$

Where the edges c, d, e are defined in figure 2 in order. Since we are talking about one dimensional graphs equation A.3 is automatically satisfied and we have to check only equation A.2 is satisfied, that is we have to verify that the following diagram is commutative.

$$\begin{tikzcd} \Gamma^{i(b)} \arrow[r, "\widehat{\phi^{i,a}_{i(b)}}"] \arrow[d, "\psi_b"'] & \Gamma^{l^{i,a}(i(b))} \arrow[d, "Ad(\widehat{\delta^{i,a}_b}) \circ \psi_{l^{i,a}(b)}"] \\ \Gamma^{t(b)} \arrow[r, "\widehat{\phi^{i,a}_{t(b)}}"] & \Gamma^{l^{i,a}(t(b))} \end{tikzcd}$$

Using the definitions in equations 3.1 and 3.2 and equation A.1 we see that the diagram is indeed commutative. We have thus defined a morphism of

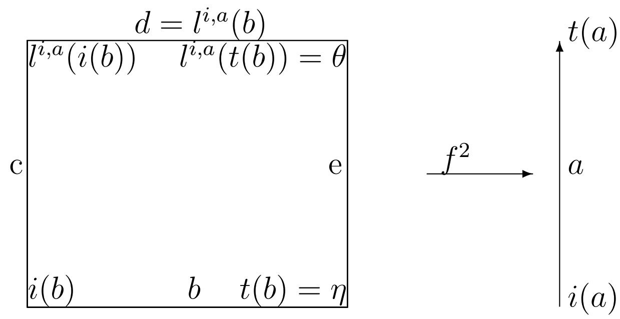

Figure 2: A typical cell in $(f^2)^{-1}(a)$

graphs of groups $\widehat{\Xi^i_a} : W^i_\epsilon \to W^i_\sigma$. The only task left is to verify that The morphism just defined is a covering morphism (as in definition A.6) of graphs of groups. We have to chekc two points

- **Every $\widehat{\phi^{i,a}_\eta} = \psi_{[\eta, l^{i,a}(\eta)]}$ is by equation 3.1 and by the definitions in section A.2 an injective homomorphism of groups.**
- **We will show that the link condition (definition 2.1) on $\Gamma(X)$ implies that: $Lk_{W^i_\epsilon}(\tilde{\eta}) \cong Lk_{W^i_\sigma}(l^{i,a}(\tilde{\eta}))$**

  Let us denote $\theta \stackrel{\text{def}}{=} l^{i,a}(\eta) \in V(X)$ this is a vertex of X and the link condition implies that $\mathrm{Lk}_{G(X)}(\tilde{\theta}) = K_{m,n}$ is a full bipartite graph. We can divide the vertices of the barycentric subdivision of this link to three disjoint sets $V(\mathrm{Lk}_{G(X)}(\tilde{\theta})) = V \cup H \cup S$ corresponding to the vertical edges, the horizontal edges and the squares in the star of θ respectively. An explicit statement of the link condition is that every $s \in S$ is an initial point of exactly two edges $v_s, h_s \in E(\mathrm{Lk}_{G(X)}(\tilde{\theta}))$ with $t(v_s) \in V$ and $t(h_s) \in H$, and the map

$$\begin{aligned} S &\to V \times H \\ s &\to (t(v_s), t(h_s)) \end{aligned} \tag{3.3}$$

  is a bijection.

  After choosing $v \in V$ we have a bijection:

$$\begin{aligned} C \stackrel{\text{def}}{=} \{s : t(v_s) = v\} &\to H \\ s &\to t(h_s) \end{aligned} \tag{3.4}$$

We will identify $C \cong \mathrm{Lk}_{W^i_\epsilon}(\tilde{\eta})$ and $H \cong \mathrm{Lk}_{W^i_\sigma}(\tilde{\theta})$ and we will also identify the isomorphism inequation 3.4 as the isomorphism induced by $\widehat{\Xi^i_a}$ on the links this will finish the proof.

We refer to figure 2 and to the explicit construction of the link in section A.11 for the discussion below. Every choice of $v \in V$ will give the desired result but for the sake of simplicity we choose $v \overset{\text{def}}{=} (\psi_e(\Gamma_{i(e)}), e)$ By equation A.29 we have

$$C = \{s : t(v_s) = v\} \cong \{(\psi_e(h)\gamma_{e,b}\psi_{eb}(\Gamma_{i(b)}), eb) : t(b) = \eta; h \in \Gamma_{i(e)}\} \tag{3.5}$$

We establish the bijection with $\mathrm{Lk}_{W^i_\epsilon}(\tilde{\eta})$ in the following way. A vertex in the link $\mathrm{Lk}_{W^i_\epsilon}(\tilde{\eta})$ corresponds to a single edge in the link $\mathrm{Lk}_{G(X)}(\tilde{\theta})$ with initial point $(\psi_e(h)\gamma_{e,b}\psi_{eb}(\Gamma_{i(b)}), eb)$ and terminal point v. This gives rise to the following bijection:

$$(\psi_e(h)\gamma_{e,b}\psi_{eb}(\Gamma_{i(b)}), eb) \to (H\psi_b(\Gamma_{i(b)}), b) \tag{3.6}$$

An obvious bijection exists between $\mathrm{Lk}_{W^i_\sigma}(\tilde{\theta})$ and H associating the two element (with the same name)

$$(g\psi_d(\Gamma_{i(d)}), d) \to (g\psi_d(\Gamma_{i(d)}), d) \tag{3.7}$$

Now we wish to obtain an explicit formula for the bijection in equation 3.4. given an element $(\psi_e(h)\gamma_{e,b}\psi_{dc}(\Gamma_{i(c)}), dc) \in C$ there is a single pair of composable edges c-vertical and d-horizontal in $E(X)$ with $eb = dc$ and we have:

$$\begin{aligned}(\psi_e(h)\gamma_{e,b}\psi_{eb}(\Gamma_{i(b)}), eb) &= (\psi_e(h)\gamma_{e,b}\psi_{dc}(\Gamma_{i(c)}), dc)\\ &= (\psi_e(h)\gamma_{e,b}(\gamma_{d,c})^{-1}\gamma_{d,c}\psi_{dc}(\Gamma_{i(c)}), dc)\\ &= (\psi_e(h)\widehat{\delta^{i,a}_b}\gamma_{d,c}\psi_{dc}(\Gamma_{i(c)}), dc)\end{aligned} \tag{3.8}$$

Using the identifications in equation 3.6 and equation 3.7 we obtain from 3.4 a bijection:

$$\begin{aligned}Lk_{W^i_\epsilon}(\tilde{\eta}) &\to Lk_{W^i_\sigma}(\widetilde{\widehat{l^{i,a}(\eta)}})\\ (h\psi_b(\Gamma_{i(b)}), b) &\to (\psi_e(h)\widehat{\delta^{i,a}_b}\psi_d(\Gamma_{i(d)}), d)\end{aligned} \tag{3.9}$$

A careful investigation of equations A.32, A.33 and of section A.11 Shows that this is indeed the morphism induced by $\widehat{\Xi^i_a}$. □

After establishing $\widehat{\Xi^i_a}$ as a covering morphism of graphs of groups. theorem A.5 gives us an injective homomorphism: $(\widehat{\delta\Xi^i_a})_{\tau^i_\epsilon} : \widehat{H^i_\epsilon} = \pi_1(\widehat{H^i_\epsilon}(W^i_\epsilon), \tau^i_\epsilon) \to \pi_1(\widehat{H^i_\sigma}(W^i_\sigma), l^{i,a}(\tau^i_\epsilon))$. Choose an arbitrary path from τ^i_σ to $l^{a,i}(\tau^i_\epsilon)$

$$\alpha_a \in \pi_1(\widehat{H^i_\sigma}(W^i_\sigma), \tau^i_\sigma, l^{i,a}(\tau^i_\epsilon)) \tag{3.10}$$

so that $Ad(\alpha_a) : \pi_1(\widehat{H^i_\sigma}(W^i_\sigma), l^{i,a}(\tau^i_\epsilon)) \to \pi_1(\widehat{H^i_\sigma}(W^i_\sigma), \tau^i_\sigma)$ is an isomorphism. Composing the two morphisms we obtain an injective homomorphism:

$$\widehat{\xi^i_a} \stackrel{\text{def}}{=} Ad(\alpha) \circ (\widehat{\delta\Xi^i_a})_{\tau^i_\epsilon} : \widehat{H^i_\epsilon} \to \widehat{H^i_\sigma} \tag{3.11}$$

We are now able to define a new graph of groups structure on X^i

Definition 3.5. *We define a graph of group structure on X^i*

$$\widehat{\Gamma^i}(X^i) \stackrel{\text{def}}{=} (X^i, \widehat{H^i_\sigma}, \widehat{\xi^i_a}) \tag{3.12}$$

where $\widehat{H^i_\sigma}$ is defined in definition 3.2(4) and $\widehat{\xi^i_a}$ is defined in equation 3.11.

The graph of group structure $\Gamma^i(X^i)$ is obtained from the action of Γ^i on T^i. The graph of group structure $\widehat{\Gamma^i}(X^i)$ is constructed from the complex of groups $\Gamma(X)$ and from the projection map $f^i : X \to X^i$. Theoretically the first construction is much simpler, but in many cases it will be much easier to calculate the second. Our main theorem for this section will show an isomorphism of the effective quotients of the two graphs of groups.

As in section A.5 the complex of groups structure on X (resp X^i) constructed from action of the group Γ (resp Γ^i) on Δ (resp T^i) is determined up to an arbitrary choice of liftings (see equation A.6 and equation A.7)

$$\{\tilde{\sigma} \in \Delta\}_{\sigma \in V(X)} \quad (\text{resp} :\{\overline{\sigma} \in T^i\}_{\sigma \in V(X^i)}) \tag{3.13}$$

and elements of Γ (resp Γ^i)

$$\{k_a \in \Gamma\}_{a \in E(X)} \quad (\text{resp} :\{m_a \in \Gamma^i\}_{a \in EX^i}) \tag{3.14}$$

Satisfying $k_a(t(\tilde{a})) = \widetilde{t(a)}$ (resp $m_a(t(\overline{a})) = \overline{t(a)}$). making a different choice changes $\Gamma(X)$ (resp $\Gamma^i(X^i)$) by a co-boundary (i.e. gives a different graph of group structure on X (resp X^i) isomorphic to $\Gamma(X)$ (resp $\Gamma^i(X^i)$) over the identity $id : X \to X$ (resp $id : X^i \to X^i$)). We will therefore feel free, all through the discussion, to change the choices mentioned above at our convenience. We introduce the following new notations:

Definition 3.6. *We introduce the following notations:*

- *For* $\sigma \in X^i$ *let*
$$S^i_\sigma \stackrel{\text{def}}{=} (p^i)^{-1} \circ p^i(\widetilde{\tau^i_\sigma}) \subset \Delta \tag{3.15}$$
be a tree, isomorphic to T^{3-i} *embedded in* Δ.

- *define*
$$\Lambda^i_\sigma \stackrel{\text{def}}{=} \{\gamma \in \Gamma | \gamma S^i_\sigma = S^i_\sigma\} < \Gamma \tag{3.16}$$

Lemma 3.7. *Let* $x, y \in V(S^i_\sigma)$ *be two (not necessarily distinct) vertices then* $\{\gamma \in \Gamma : \gamma x = y\} < \Lambda^i_\sigma$

Proof. Obvious. □

Proposition 3.8. *The following hold:*

1. *There exist a group isomorphism and an equivariant isomorphism of trees making the following diagram commutative.*

$$\begin{tikzcd}
\Lambda^i_\sigma \arrow[r] \arrow[d] & \widehat{H^i_\sigma} \arrow[d] \\
S^i_\sigma \arrow[r] & \widetilde{\widehat{H^i_\sigma}(W^i_\sigma)}
\end{tikzcd}$$

2. *For every* $a \in E(X^i)$ *with* $i(a) = \epsilon \quad t(a) = \sigma$ *the commutative diagrams in* 1 *associated with* σ *and with* ϵ *are compatible. In the sense that there exist a commutative diagram of actions:*

$$\begin{tikzcd}
\Lambda^i_\epsilon \arrow[rrr] \arrow[dr, "Ad(\chi(a)\circ k)"] \arrow[ddd] & & & \widehat{H^i_\epsilon} \arrow[dl, "(\widehat{\xi^i_a})_{\tau^i_\epsilon}"'] \arrow[ddd] \\
& \Lambda^i_\sigma \arrow[r] \arrow[d] & \widehat{H^i_\sigma} \arrow[d] & \\
& S^i_\sigma \arrow[r] & \widetilde{\widehat{H^i_\sigma}(W^i_\sigma)} & \\
S^i_\epsilon \arrow[ur, "\chi(a)\circ k\circ\widetilde{l^{i,a}}"'] \arrow[rrr] & & & \widetilde{\widehat{H^i_\epsilon}(W^i_\epsilon)} \arrow[ul, "\widetilde{\widehat{\xi^i_a}}"]
\end{tikzcd}$$

Where $\widetilde{l^{i,a}}$ *is obtained by lifting the strip* $(f^i)^{-1}(a)$ *of lemma 3.4 to the universal cover,* $k = k_{\widetilde{\tau^i_a}} = k_{[\tau^i_\epsilon, l^{i,a}(\tau^i_\epsilon)]}$ *is the element mentioned in equation 3.14 satisfying* $k(t(\widetilde{\tau^i_a})) = \widetilde{t(\tau^i_a)}$ *and* $\chi(a) \in \Lambda^i_\sigma < \Gamma$ *is the element of* Λ^i_σ *corresponding to the path* $\alpha(a)$ *of equation 3.10 according to lemma 2.5.*

Proof. We will choose the elements $\{\tilde{\sigma}\}_{\sigma \in V(X)}$ and $\{k_a\}_{a \in E(X)}$ so as to obtain the above results. First observe that from the commutative diagram below we obtain the existence of the morphism h^i_σ embedding the graph $\Lambda^i_\sigma \backslash S^i_\sigma$ into X.

$$\begin{tikzcd}
\Lambda^i_\sigma \arrow[r] \arrow[d] & \Gamma \arrow[d] \\
S^i_\sigma \arrow[r] \arrow[d] & \Delta \arrow[d] \\
\Lambda^i_\sigma | S^i_\sigma \arrow[r, dashed, "\exists h^i_\sigma"] & \Gamma | \Delta
\end{tikzcd}$$

In fact h^i_σ is an isomorphism of $\Lambda^i_\sigma \backslash S^i_\sigma$ onto $W^i_\sigma \subset X$. Surjectivity follows from local surjectivity and W^i_σ being connected (lemma 3.1). Injectivity follows from lemma 3.7: If two points of S^i_σ are identified $\bmod\, \Gamma$ then they are also identified mod Λ^i_σ.

Now from surjectivity every $\eta \in V(X)$ is in the image of some S^i_σ. We may therefore assume that the choices $\tilde{\eta}$ in equation 3.13 have been made so that $\tilde{\eta} \in S^i_\sigma$ for every η and some $\sigma = \sigma(\eta) \in X^i$. Lemma 3.7 now implies two things

- $k_a \in \Lambda^i_\sigma$ for every $a \in E(W^i_\sigma)$.
- $\mathrm{Stab}_\Gamma(\tilde{\eta}) = \mathrm{Stab}_{\Lambda^i_\sigma}(\tilde{\eta})$.

It is obvious by all of the above that the graph of group structure of $\Gamma \backslash\backslash \Delta$ restricted to W^i_σ and the graph of group structure on $\Lambda^i_\sigma \backslash\backslash S^i_\sigma$ are exactly isomorphic. This proves part 1.

Now for part 2 of the claim look at the following diagram:

$$\begin{tikzcd}
\Lambda^i_\epsilon \arrow[r, "Ad(k)"] \arrow[d] & \Lambda^i_\sigma \arrow[d] \\
S^i_\epsilon \arrow[r, "k \circ \widetilde{l^{a,i}}"] \arrow[d] & S^i_\sigma \arrow[d] \\
\widehat{H^i_\epsilon}(W^i_\epsilon) \arrow[r, dashed, "\exists \Xi^i_a"] & \pi_1(\widehat{H^i_\sigma}(W^i_\sigma, l^{i,a}(\tau^i_\epsilon)))
\end{tikzcd}$$

By section A.6 (or [Hae91, section2.3], [Bas93, section 4]) The morphism of actions induces a morphism of graphs of groups Ξ_a^i. Like in the previous section we wish to show that we can make the right choices so as to obtain: $\Xi_a^i = \widehat{\Xi_a^i}$. First observe that, since $k \in \Gamma$, the topological morphism induced by the commutative diagram above is equal to $l^{i,a}$. Now we make the choices of equation A.11 in order to define $\Xi_a^i = (l^{i,a}, \phi_\eta^{i,a}, \delta_b^{i,a})$ as a homomorphism of graphs of groups. We choose $\{\lambda_\eta \in \Lambda_\sigma^i\}_{\eta \in V(W_\epsilon^i)}$ satisfying $\lambda_\eta \circ k \circ \widetilde{l^{i,a}}(\tilde{\eta}) = \widetilde{l^{i,a}(\eta)}$ We define:

$$\lambda_\eta \stackrel{\text{def}}{=} k_{[\eta, l^{i,a}(\eta)]} k^{-1} \tag{3.17}$$

So that $\lambda_\eta \circ k = k_{[\eta, l^{i,a}(\eta)]}$. By our choice of liftings $\widetilde{l^{i,a}(\eta)}, \widetilde{l^{i,a}(\tau_\epsilon^i)} \in S_\sigma^i$ and $\widetilde{\eta}, \widetilde{\tau_\epsilon^i}) \in S_\epsilon^i$ so indeed $\lambda_\eta \in \Lambda_\sigma^i$. Using equation A.12 we define:

$$\phi_\eta^{i,a} = \mathrm{Ad}(\lambda_\eta) \circ \mathrm{Ad}(k) = \mathrm{Ad}(k_{[\eta, l^{i,a}(\eta)]}) = \widehat{\phi_\eta^{i,a}} \tag{3.18}$$

Where the last equality is just definition in equation 3.1. Using equation A.13 $\delta_b^{i,a}$ is given by:

$$\delta_b^{i,a} = \lambda_{t(b)} \mathrm{Ad}(k)(k_b) \lambda_{i(b)}^{-1} k_{l^{i,a}(b)}^{-1} = k_{[t(b), l^{i,a}(t(b))]} k_b k_{[i(b), l^{i,a}(i(b))]}^{-1} k_{l^{i,a}(b)}^{-1} = \widehat{\delta_b^{i,a}} \tag{3.19}$$

where the last equality holds because it is compatible with the commutative diagram on page 11, after using equation 3.1 A.9 and 3.14. Composing everything with $\chi(a)$ we obtain exactly the desired commutative diagram. □

Theorem 3.9. *The two graph of group structures on X. are isomorphic*

$$\Gamma^i(X^i) = (X^i, H_\sigma^i, \xi_a^i) = \widehat{\Gamma}^i(X^i) = (X^i, \widehat{H_\sigma^i}, \widehat{\xi_a^i}) \tag{3.20}$$

Proof. Now we use the freedom to choose the elements $\overline{\sigma}$ and m_a defined in equations 3.13 and 3.14, and used to construct the graph of group structure $\Gamma^i(X^i)$. Let us choose:

$$\overline{\sigma} = p^i(\widetilde{\tau_\sigma^i}) \tag{3.21}$$

$$m_a = \phi^i(\chi(a)k) \tag{3.22}$$

By equation A.8 for every $\sigma \in V(X^i)$

$$H_\sigma^i = Stab_{\Gamma^i}(\overline{\sigma}) = \phi^i(\Lambda_\sigma^i) = \widehat{H_\sigma^i} \tag{3.23}$$

Where the last equality is true (only up to taking an effective quotient) by part 1 of proposition 3.8. By equation A.9 for every $a \in E(X^i)$ the homomorphism ξ_a^i is given by:

$$\xi_a^i(\phi^i(\lambda)) = m_a\phi^i(\lambda)m_a^{-1} = \phi^i(\mathrm{Ad}(\chi(a)k)(\lambda)) = \phi^i(\xi_a^i(\lambda)) \tag{3.24}$$

Again the effect of ϕ^i here is similar to the effect of taking an effective quotient of $\widehat{\Gamma}^i(X^i)$. The last equality is true by part 2 of proposition 3.8. This completes the proof of the theorem. □

4 A Dichotomy Theorem

In this section we deal with the situation in which $\Gamma < \mathrm{Aut}(\Delta)$ is an irreducible locally primitive lattice. where

Definition 4.1. *A lattice $\Gamma < \mathrm{Aut}(\Delta)$ is called locally primitive if the action of the projections Γ^i on the trees T^i is locally primitive, in the sense that for every vertex in T^i it's stabilizer acts primitively on the link of the vertex.*

The notion of an irreducible locally primitive lattice turns out to be a good analog of an irreducible lattice in a semi simple lie group (see [BM00, BMZ]). There are many results concerning properties of homomorphic images of irreducible lattices. Margulis' normal subgroup theorem of [Mar91] states that the image of an irreducible lattice in a connected semisimple Lie group without compact factors and with a finite center is either finite or of finite index. Burger and Mozes follow Margulis' proof and give a similar normal subgroup theorem for lattices in products of trees [BM00]. Explicitly assuming that Γ is locally infinitely transitive [3] and that $(H^i)^{(\infty)} < H^i$ is of finite index [4], They show that every homomorphic image of Γ is either finite or isomorphic to Γ.

If we assume only that Γ is locally infinitely transitive then Margulis' work implies that every proper image of Γ has Kazhdan's property (T). A similar theorem was proved lately in a more geometric method by Pansu in [Pan]. In particular every proper image of Γ in this situation has property (FA) of Serre.

[3] $\Gamma < \mathrm{Aut}(\Delta)$ is called locally infinitly transitive if $\mathrm{Stab}_{\Gamma^i}(\sigma)$ is transitive on paths without inversions of arbitrary length starting at any given vertex $\sigma \in T^i$

[4] H^i is the closure of the projection of Γ. and $(H^i)^{(\infty)}$ is the subgroup of all elements with open centralizers.

In this section we deal only with the specific normal subgroup $N \triangleleft \Gamma$ generated by all vertex stabilizers. From a geometric point of view if $\Gamma(X) = \Gamma\backslash\backslash\Delta$ is the complex of groups associated with the action then $\Gamma = \pi_1(\Gamma(X), \sigma_0)$ and $\Gamma/N = \pi_1(X, \sigma_0)$. The square complex of groups $\Gamma(X)$ admits, by construction, a universal cover Δ which is a product of two trees (and therefore satisfies the link condition (definition 2.1) by theorem 2.2). This will usually not be true for X. In fact if we assume that Γ is locally infinitely transitive we may deduce the following corollary of Margulis normal subgroup theorem:

Theorem 4.2. *If $\Gamma < \mathrm{Aut}(\Delta)$ is locally infinitely transitive, and if $X = \Gamma\backslash\Delta$ is covered by a product of trees. Then one of the following holds:*

- *$N = \Gamma$, X is contractible and Γ is generated by finite groups.*
- *Γ is torsion free and acts freely on Δ.*

Proof. Assume $N \subsetneqq \Gamma$ is a proper subgroup. Then Γ/N has property (FA). Γ/N acts on its universal cover $\tilde{X} = Y_1 \times Y_2$ and thus acts on both trees Y_i so both Y_i must be finite and Γ/N must be a finite group. $\tilde{X}$ is a two dimensional contractible square complex so $X \simeq K(\Gamma/N, 1)$ and Γ/N has a cohomological dimension $\leq$ 2. A finite group with finite cohomological dimension must be trivial, and the theorem follows. □

Remark Observe that we can not use the theorem of Burger and Mozes because we do not assume $(H^i)^{(\infty)} < H_i$ is of finite index.

Proposition 4.3. *Given a group $\Gamma < \mathrm{Aut}(\Delta)$ and a vertex σ of Δ let us denote by $\Lambda = \Lambda(\sigma) \rightsquigarrow K_{A,B}$ the action of a finite group on a full bipartite graph which corresponds to the actions of $\mathrm{Stab}_\Gamma(\sigma)$ on the $\mathrm{Lk}_\Delta(\sigma)$. The following conditions are equivalent for Γ:*

1. *$X = \Gamma\backslash\Delta$ is covered by a product of two trees.*
2. *X satisfies the link condition.*
3. *The action of Γ_σ on the link of a vertex σ denoted $\Lambda(\sigma) \rightsquigarrow K_{A,B}$ (where $K_{A,B}$ is the full bipartite graph) satisfies the following condition. For every $f_1, f_2 \in \Lambda(\sigma); a \in A; b \in B$. There exists $f \in \Lambda(\sigma)$ such that $f(a) = f_1(a); f(b) = f_2(b)$.*
4. *X is locally CAT(0).*

5. $\Lambda(\sigma)\backslash B \cong \mathrm{Stab}_{\Lambda(\sigma)}(a)\backslash B$ *for every* $a \in A$ *where* $\mathrm{Stab}_{\Lambda(\sigma)}(a)$ *is the stabilizer of* $a \in A$.

6. *for every* $a \in A;\ b \in B$ *we can write* $\Lambda(\sigma)$ *as a product* $\Lambda(\sigma) = \mathrm{Stab}_{\Lambda(\sigma)}(a)\,\mathrm{Stab}_{\Lambda(\sigma)}(b)$

Proof.

1 $\Leftrightarrow$ 2 Is just theorem 2.2.

2 $\Leftrightarrow$ 3 3 is just an explicit statement of 2.

2 $\Leftrightarrow$ 4 An Euclidean locally compact polyhedral complex is locally CAT(0) if and only if the link of every vertex is CAT(1), but a quotient of a full bipartite graph is CAT(1) if and only if it is itself a full bipartite graph.

3 $\Rightarrow$ 5 Use 3 with $f_1 = 1$ and $f_2 \in \Lambda(\sigma)$.

5 $\Rightarrow$ 3 Find by 5 an element $g \in \mathrm{Stab}_{\Lambda(\sigma)}(a)$ such that $g(b) = f_1^{-1}f_2(b)$. Then $f_1g(a) = f_1(a)$ and $f_1g(b) = f_1f_1^{-1}f_2(b) = f_2(b)$.

5 $\Rightarrow$ 6 Let $f \in \Lambda(\sigma); a \in A; b \in B$ be given. By 5 there is an element $g \in \mathrm{Stab}_{\Lambda(\sigma)}(a)$ such that $g(b) = f(b)$. Now write $f = g(g^{-1}f)$ where $g \in \mathrm{Stab}_{\Lambda(\sigma)}(a)$ and $g^{-1}f \in \mathrm{Stab}_{\Lambda(\sigma)}(b)$.

6 $\Rightarrow$ 5 Let $f \in \Lambda(\sigma), a \in A, b \in B$ be given. Write $f = gh$ where $g \in \mathrm{Stab}_{\Lambda(\sigma)}(a)$ and $h \in \mathrm{Stab}_{\Lambda(\sigma)}(b)$ then $f(b) = gh(b) = g(b)$. The $\Lambda(\sigma)$ and the $\mathrm{Stab}_{\Lambda(\sigma)}(a)$ orbits in B are, therefore, the same.

Definition 4.4. *If* $\Gamma < \mathrm{Aut}\,\Delta$ *satisfies any (and hence all) of the conditions specified in proposition 4.3 we say that* Γ *satisfies a* strong link condition.

Our purpose in this section is to give a generalization of theorem 4.2 using the Machinery developed in section 3.

Theorem 4.5. *Let* $\Gamma < \mathrm{Aut}(\Delta)$ *be an irreducible locally primitive lattice satisfying a strong link condition whose projections act without inversion. Let* $N \triangleleft \Gamma$ *be the (normal) subgroup generated by all vertex stabilizers. Then the following dichotomy holds for* $X \stackrel{\mathrm{def}}{=} \Gamma\backslash\Delta$.

1. either X *is a single square. In which case* Γ *is generated by finite groups and* $\Gamma = N$

2. or $X = \Gamma(X)$ *in which case* Γ *is torsion free and* N *is trivial*

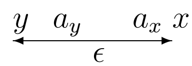

Figure 3: An edge of groups.

4.1 Some Notation and preliminary remarks

We will use the notation of section 3. Under the conditions of theorem 4.5, X^i is an edge of groups (drawn in figure 3) given by $\Gamma^i(X^i) = (X^i, H^i_\sigma, \xi^i_a)$. By section 3, $H^i_\sigma \cong \pi_1(H^i_\sigma(W^i_\sigma), \tau^i_\sigma)$[5]. By proposition 2.3, we may write $H^i_\sigma = F^i_\sigma \ltimes N^i_\sigma$ where $F^i_\sigma \stackrel{\mathrm{def}}{=} \pi_1(W^i_\sigma, \tau^i_\sigma)$ and $N^i_\sigma \stackrel{\mathrm{def}}{=} \langle \Gamma_\eta | \eta \in V(W^i_\sigma) \rangle$. By proposition 2.4, $\xi^i_{a_\sigma}(N^i_\epsilon) < N^i_\sigma$ And the morphism induced from $\xi^i_{a_\sigma}$ on the quotient $\overline{\xi^i_{a_\sigma}} : F^i_\epsilon \to F^i_\sigma$ is just the morphism $(l^{i,a_\sigma})_*$. X satisfies the link condition so the natural graph morphism l^{i,a_σ} is a covering and the group morphism $(l^{i,a_\sigma})_*$ is an injection. This means that:

$$\xi^i_{a_\sigma}(F^i_\epsilon) \cap N^i_\sigma = \emptyset \tag{4.1}$$

The following is a direct result of primitivity.

Proposition 4.6. *$\xi^i_{a_\sigma}(H^i_\epsilon) \lneqq H^i_\sigma$ is a maximal subgroup.*

Proof. The action of Γ^i on T^i is isomorphic to the action of the amalgam associated with the edge of groups $\Gamma^i(X^i)$ on its universal cover. In order to obtain the local action on the link of a vertex we must divide both $\xi^i_{a_\sigma}(H^i_\epsilon)$ and H^i_σ by the normal subgroup $(H^i_\sigma)_1$ stabilizing a sphere of radius one. By Our assumption of Local primitivity $\xi^i_{a_\sigma}(H^i_\epsilon)/(H^i_\sigma)_1 \lneqq H^i_\sigma/(H^i_\sigma)_1$ is maximal. The result now follows from the correspondence between subgroups of $H^i_\sigma/(H^i_\sigma)_1$ and subgroups of H^i_σ containing $(H^i_\sigma)_1$. $\square$

By equation 4.1 the product of the groups $\xi^i_{a_\sigma}(F^i_\epsilon)N^i_\sigma = \xi^i_{a_\sigma}(F^i_\epsilon) \ltimes N^i_\sigma$ is a semi-direct product. we may write now:

$$\xi^i_{a_\sigma}(F^i_\epsilon) \ltimes \xi^i_{a_\sigma}(N^i_\epsilon) \stackrel{1}{\leq} \xi^i_{a_\sigma}(F^i_\epsilon) \ltimes N^i_\sigma \stackrel{2}{\leq} F^i_\sigma \ltimes N^i_\sigma \tag{4.2}$$

By proposition 4.6 we must have equality in one of the two inclusions of equation 4.2. Let us examine the meaning of each of the two cases.

[5]We will drop all the hats from now on. This is legitimate by to theorem 3.9

equality in 1 This implies $\xi^i_{a_\sigma}(N^i_\epsilon) = N^i_\sigma$ in particular $\xi^i_{a_\sigma}(N^i_\epsilon) \triangleleft H^i_\sigma$ and therefore acts trivially on $B_{T^i}(\overline{\sigma}, 1)$ The one ball around lifting of σ

equality in 2 this implies that the *covering* morphism: $l^{i,a_\sigma} : W^i_\epsilon \to W^i_\sigma$ induces an isomorphism on the fundamental groups. and is therefore an isomorphism of graphs. We then observe that $(f^i)^{-1}(a_\sigma)$ is a product of a graph and an edge for example $(f^2)^{-1}(a_\sigma) \cong W^2_\sigma \times (a_\sigma) \cong W^2_\epsilon \times (a_\sigma)$

Equality in 1 or in 2 holds in any one of the four "sites": $i \in \{1, 2\}$ and $\sigma \in \{x, y\}$

4.2 Proof of theorem 4.5

We will show that the two cases in which 1 or 2 hold in all four sites correspond to the two cases stated in the theorem. Then we will show that all other 14 possibilities are impossible. We proceed in four steps.

4.2.1 step1

if equality in 1 holds for all four sites $\xi^i_{a_\sigma}(N^i_\epsilon) \triangleleft H^i_\sigma$ in all four sites. Then N^i_ϵ, being normal in H^i_x and in H^i_y, acts trivially on T^i. N^i_σ also acts trivially on T^i because of the equality $\xi^i_{a_\sigma}(N^i_\epsilon) = N^i_\sigma$ $\forall \sigma \in \{x, y\}$. Finally the normal subgroup generated by all stabilizers $N \stackrel{\text{def}}{=} \langle N^i_\sigma : \sigma \in \{x, y\}\rangle$ acts trivially on T^i. This is true for both $i \in 1, 2$ so, after passing to the effective quotient, $\Gamma(X)$ is actually a complex of trivial groups and Γ - its fundamental group acts freely and discreetly on Δ and is therefor torsion free. This situation corresponds to case 2 in the statement of the theorem.

4.2.2 step2

If equality in 2 holds for all four sites then

$$X = (f^2)^{-1}(X^2) \cong W^2_\epsilon \times (X^2) \cong (f^1)^{-1}(X^1) \cong X^1 \times W^1_\epsilon \cong X^1 \times X^2 \tag{4.3}$$

Thus X is a square and $\Gamma(X)$ is a square of groups. This situation corresponds to case 1 in the statement of the theorem.

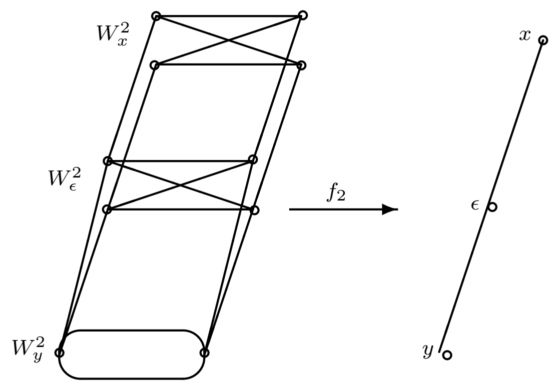

Figure 4: Illustration for step 4

4.2.3 step3

Assume first that equality in 2 holds in two sites of different sides. We may assume without loss of generality that the sites are ($i = 1; \sigma = x$ and $i = 2; \sigma = x$), so $(f^1)^{-1}(a_x) \cap (f^2)^{-1}(a_x)$ is a square. Since $(f^i)^{-1}(e)$ covers $(f^i)^{-1}(y)$ for $i \in \{x, y\}$ we must have X-a square and we are back to the case of step 4.2.2.

4.2.4 step4

Assume that equality in 2 holds for one site (say $i = 2, \sigma = x$). Then either we are back to the case of step 4.2.2 or, by step 4.2.3, equality in 1 holds for $i = 1; \sigma \in \{x, y\}$. Reasoning as in step 4.2.1 we conclude that $N \triangleleft \Gamma$, the subgroup generated by all vertex stabilizers, acts trivially on T^1. The action morphism $\Gamma \to \mathrm{Aut}\, T^1$ then splits via the quotient $\Gamma \to \Gamma/N \cong \pi_1(X, \sigma_0) \to \mathrm{Aut}(T^1)$. Equality in 2 holds for ($i = 2; \sigma = x$) so $(f^2)^{-1}(a_x) = W_\epsilon^2 \times a_x = W_x^2 \times a_x$. $l^{2,a_y} : W_\epsilon^2 \to W_y^2$ is a covering morphism of graphs (The situation is drawn in figure 1). By the Siefert van Kampen theorem

$$\pi_1(X, -) = \pi_1(W_x^2, -) *_{\pi_1(W_\epsilon^2, -)} \pi_1(W_y^2, -) = \pi_1(W_y^2, -) \tag{4.4}$$

And the action on T^1 is isomorphic to the action of $\pi_1(W_y^2, -)$ on the universal cover $\widetilde{W_y^2}$ of the graph of groups (notice that $(\widetilde{W_2^y}) \cong (\widetilde{H_y^2(W_2^y)}) \cong T^1$ because all vertex stabilizers act trivially on T^1). This action is discreet - contradicting the irreducibility of Γ and proving the theorem. $\square$

5 A Two Dimensional Analog of the Goldshmidt Sims Conjecture

The Goldshmidt Sims Conjecture Asserts that, up to isomorphism of amalgams, there is only a finite number of effective primitive amalgams $A *_{\Xi} B$ of finite groups such that the indices $[A : \Xi] = p$ and $[B : \Xi] = q$ are given. If p and q are prime the amalgam is automatically locally primitive so in this case the conjecture claims there is a finite number of effective amalgams. In terms of the Bass Serre theory the conjecture may be stated in the following terms: given an edge indexed graph, consisting of just one edge. there is a finite number of finite effective groupings such that the corresponding fundamental group acts locally primitively on the universal cover. In [Gol80] Goldshmidt proves this conjecture in the case $p = q = 3$. (see also [Fan86a, DGS85]) but a general solution to this problem seems quite difficult. The Thompson and Wielandt theorem (see theorem 1.4, [Fan86b], [BCN80]) reduces the general situation to a discussion of l-elementary amalgams. Specifically it is shown that for a discreet locally primitive subgroup the stabilizers of balls of radius 2 on the tree are l-groups where $l < \min\{p, q\}$ is a prime.

In this note I present a very short proof for a two dimensional version of the Goldshmidt Sims conjecture.

Theorem 5.1. *Let X be square, $\{q_i\}_{i=1\ldots2}$ primes Then there is only a finite number of effective groupings $\Gamma(X)$ of X such that:*

- *the universal cover $\Delta \stackrel{\text{def}}{=} (\widetilde{\Gamma(X), \sigma_0}) = T^1 \times T^2$ is a product of two regular trees of valence q_1, q_2.*
- *the stabilizers act effectively on both trees.*

The requirement that the stabilizers act effectively on both trees is a requirement of irreducibility. We will show in theorem 5.2 that this requirement holds, in particular, in the case that $\Gamma \stackrel{\text{def}}{=} \pi_1(\Gamma(X), \sigma_0)$ is an irreducible, locally primitive lattice in $Aut(\Delta)$. A proof of the theorem without this irreducibility assumption is equivalent to proving the one dimensional conjecture above in the case where $p = q$ is prime, because given an edge of groups we can construct from it a square of groups by taking a product.

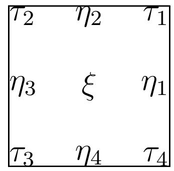

Figure 5: A square of groups

5.1 proof of theorem

Let us assume that we are given a finite effective grouping $\Gamma(X) = (X, \Gamma_\sigma, \psi_a, g_{a,b})$ of the given indexed square such that the universal cover $\Delta = \widetilde{\Gamma(X)} = T^1 \times T^2$ is a product of two regular trees of prime valence and such that the stabilizers act effectively on both trees. It will be enough if we give a bound $M = M(q_1, q_2)$ depending only on the two primes q_i for the order of the stabilizer of the square Γ_ξ.

We may assume without loss of generality that $q_2 \leq q_1$. A short examination will show that, after modifying $\Gamma(X)$ by a co-boundary (see section A.4), we may assume that all $g_{a,b}$ are trivial. We can now think of our square both as the quotient of Δ by the action of $\Gamma = \pi_1(\Gamma(X), \sigma_0)$ and as a fundamental domain for this action. We will adopt the second approach and think of the square as one of the squares in Δ. This square is drawn in figure 5. Let $p^i : \Delta \longrightarrow T^i$ be the projections. This situation is drawn in figure 5.

Let us denote by $\Gamma^i_\sigma = \phi^i(\Gamma_\sigma)$ the image of Γ_σ acting on T^i and by $\Gamma^i_\sigma(n)$ the subgroup of Γ^i_σ stabilizing the ball $B_{T^i}(p^i(\sigma), n)$ of radius n around the vertex or edge in T^i associated with σ. Our second assumption says that $\Gamma_\sigma \cong \Gamma^i_\sigma$ for all σ.

We claim that $\Gamma^2_\xi(2) = \langle e \rangle$ so that there exists an embedding $\Gamma_\xi = \Gamma^2_\xi \hookrightarrow \mathrm{Aut}(B_{T^2}(p^2(\xi), 2))$ into a finite group giving the desired bound on the order of Γ_ξ.

$$|\Gamma_\xi| \leq \left|\mathrm{Aut}(B_{T^2}(p^2(\xi), 2))\right| = (q_2 - 1)!^{2q_2} \tag{5.1}$$

Let us assume the contrary: $\Gamma^2_\xi(2) \neq \langle e \rangle$. Look at the subgroup of Γ stabilizing $p^1(\xi)$ namely $\Gamma_{p^1(\xi)} \overset{\mathrm{def}}{=} (\phi^1)^{-1}(Stab_{\Gamma^1}(p^1(\xi))) \cong \Gamma_{\eta_2} *_{\Gamma_\xi} \Gamma_{\eta_4}$. The action of Γ^2_ξ on T^2 is canonically isomorphic to the action of $\Gamma^2_\xi \hookrightarrow \Gamma_\xi$ on the universal cover of this amalgam. Since this action is discreet and locally transitive

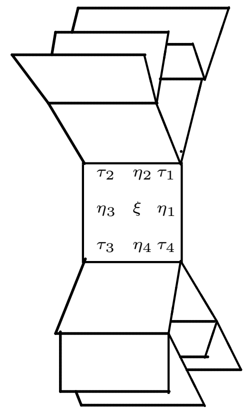

Figure 6: $\Gamma^2_{\eta_j}(2) \subset \Gamma^2_\xi(2)$

on a regular tree of prime valence it is locally primitive. Same argument holds for $\Gamma_{p^1(\eta_j)}$ for $j \in \{1, 3\}$. By the Thompson Wielandt 1.4 theorem there exist a prime $l < q_2 \leq q_1$ such that $\{\Gamma^2_\sigma(2)\}_{\sigma \in \{\xi, \tau_1, \tau_3\}}$ are all l-groups. l is the same prime for all three groups because $\Gamma^2_{\eta_j}(2) \supset \Gamma^2_\xi(2)$ for $j \in \{1, 3\}$. $\Gamma^2_\sigma(2) \triangleleft \Gamma^2_\sigma$ is a normal l-subgroup for every σ and therefore

$$\Gamma^2_\sigma(2) \subset \mathcal{O}_l(\Gamma^2_\sigma) = \bigcap_{L \in Syl_l(\Gamma^2_\sigma)} L \tag{5.2}$$

Since $[\Gamma^2_{\eta_j} : \Gamma^2_\xi] = q_1 > l$ every l-Sylow subgroup of Γ^2_ξ is also an l-Sylow subgroup of $\Gamma^2_{\eta_j}$ so equation 5.2 implies:

$$\Gamma^2_{\eta_j}(2) \subset \mathcal{O}_l(\Gamma^2_{\eta_j}) \subset \mathcal{O}_l(\Gamma^2_\xi) \subset \Gamma^2_\xi = \Gamma_\xi \tag{5.3}$$

An element of $\Gamma^2_{\eta_j}$ stabilizing both ξ and a ball of radius 2 in the vertical direction (i.e. the direction of T^2) around η_j must stabilize the ball of radius 2 in the vertical direction around ξ. So that equation 5.3 implies $\Gamma^2_{\eta_j}(2) \subset \Gamma^2_\xi(2)$. This is demonstrated in figure 6. Since $\Gamma^2_\xi(2) \subset \Gamma^2_{\eta_j}(2)$ is obvious we obtain:

$$\Gamma^2_\xi(2) = \Gamma^2_{\eta_j}(2) \quad \forall j \in \{1, 3\} \tag{5.4}$$

$\Gamma^2_{\eta_j}(2)$ is a normal subgroup of $\Gamma^2_{\eta_j}$ for every $j \in \{1, 3\}$

$$\Gamma^2_\xi(2) \triangleleft \Gamma^2_{\eta_j} \cong \Gamma_{\eta_j} \cong \Gamma^1_{\eta_j} \quad \forall j \in \{1, 3\} \tag{5.5}$$

The action of Γ^1_ξ on T^1 is exactly identical to the action of $\Gamma^1_\xi \hookrightarrow \Gamma^1_{\eta_3} *_{\Gamma^1_\xi} \Gamma^1_{\eta_1}$ on the universal cover of the associated edge of groups so by equation 5.5 $\Gamma^2_\xi(2)$ acts trivially on T^1. By our second assumption $\Gamma^2_\xi(2)$ must be a trivial subgroup of Γ_ξ and this is a contradiction. $\square$

5.2 Irreducible Locally primitive lattices

We recall that a lattice $\Gamma < \mathrm{Aut}(\Delta) = \mathrm{Aut}(T^1 \times T^2)$ is called irreducible if its projections $\phi^i(\Gamma) \subset \mathrm{Aut}(T^i)$ are non discreet. It is called locally primitive if its projections act locally primitively. Observe that if the tree has prime valence local primitivity is implied by local transitivity. It has been shown in works of Burger Mozes and Zimmer [BM00, BMZ] that from many aspects irreducible locally primitive lattices are analogous to irreducible lattices in higher rank lie groups.

Here we show that, in our setting, if the fundamental group of a square of groups is an irreducible locally primitive lattice then the assumptions of theorem 5.1 are satisfied. Specifically we show the following:

Theorem 5.2. *Let $\Gamma < \mathrm{Aut}(\Delta) = \mathrm{Aut}(T^1 \times T^2)$ be a uniform irreducible lattice in the automorphism group of the product of two regular trees. Let $\phi^i : \mathrm{Aut}(\Delta) \longrightarrow \mathrm{Aut}(T^i)$ be the projection maps. Then $\ker(\phi^i) < \Gamma$ is torsion free. In particular all stabilizers in Γ act effectively on the projections.*

We first need the following lemma due to Burger and Mozes

Lemma 5.3. *let $\Lambda < \mathrm{Aut}(T)$ be a locally primitive group acting on a regular tree T. Then every normal subgroup $N \triangleleft \Lambda$ is either torsion free or co-compact.*

Proof. see [BM00] $\square$

Proof of the theorem. The action of Γ on Δ is, by assumption, effective so every element ($\neq id$) of Γ acting trivially on T^i acts non trivially on T^{3-i}. In other words $\ker(\phi^i)$ injects under ϕ^{3-i} into a normal subgroup of $\Gamma^{3-i}(= \phi^{3-i}(\Gamma))$. By assumption Γ^{3-i} acts locally primitively on T^{3-i} so by lemma 5.3 every normal subgroup of Γ^{3-i} has to be ether torsion free or co-compact. It remains to show that $\phi^{3-i}(\ker(\phi^i))$ can not be co-compact. If it were we could have found a fundamental domain with a finite number of vertices $\{v_1, v_2, \ldots, v_n\}$ for its action on T^{3-i}. For every $\gamma \in \Gamma$ we can find now $\delta \in \ker(\phi^i)$ such that $\phi^{3-i}(\delta \circ \gamma)v_1 \in \{v_1, v_2, \ldots, v_n\}$. But $\phi^i(\gamma) = \phi^i(\delta \circ \gamma)$. If we denote by $\Lambda = \{\gamma \in \Gamma | \phi^{3-i}(\gamma)v_1 = v_1\}$ Then $\phi^i : \Gamma \longrightarrow \Gamma_i$ is still surjective even if we restrict it to a finite union of cosets of Λ. Namely choose $\{\gamma_k\}_{k=1..n}$ such that $\phi^{3-i}(\gamma_k)v_1 = v_k$ then $\phi^i : \bigcup_{k=1}^n \gamma_k\Lambda \longrightarrow \Gamma_i$ is surjective. The image by ϕ^i of each one of these cosets is discreet and the image of Γ is, by assumption, non-discreet so we obtain a contradiction. This proves the theorem. □

6 A minimal co-volume theorem

We are now able to prove a minimal co-volume theorem for lattices satisfying the strong link condition.

Theorem 6.1. *Let $q_1, q2$ be two primes. $\Delta = T^1 \times T^2 = T^{q_1} \times T^{q_2}$ be a product of the two regular trees of the given valence. Then there exists a constant $m = m(\min\{q_1, q_2\})$ such that for every $\Gamma < \mathrm{Aut}(\Delta)$ an irreducible, locally primitive lattice satisfying a strong link condition $\mathrm{Vol}\,\Gamma\backslash\,\mathrm{Aut}(\Delta) \leq m$.*

Proof. First we may assume that Γ acts without inversion on both trees by passing to a subgroup of index ≤ 4. Now by theorem 4.5 Γ is ether torsion free or admits a square as a fundamental domain. Let us normalize Harr measure so that the volume of a stabilizer of a square is taken to be 1. All torsion free lattices have a co-volume of at least 1 so taking $m < 1$ we may disregard them. Assume $\Gamma\backslash\Delta$ is a square. By theorem 5.2 the conditions for theorem 5.1 are satisfied. The later theorem finishes the proof. An explicit value for m is given by equation 5.1

$$m = \frac{1}{4(q-1)!^{2q}} \tag{6.1}$$

□

7 An infinite family of irreducible lattices.

In this section I wish to give an example for a family of irreducible uniform lattices $\Gamma_j < \mathrm{Aut}(\Delta)$ such that $\Gamma_j \lneq \Gamma_{j+1} \lneq \Gamma_{j+2} \ldots$. In particular $\lim_{j\to\infty} \mathrm{covol}(\Gamma_j) = 0$. First take $\Lambda < \mathrm{Aut}(\Delta') = \mathrm{Aut}(T^{1'} \times T^{2'})$ an irreducible torsion free lattice such that $X \stackrel{\mathrm{def}}{=} \Lambda\backslash\Delta'$ is a square complex with four vertices. We will build a family of lattices $\{\Gamma_j\}_{j=1\ldots\infty}$ by constructing a family

$$G_j \xleftarrow{\quad} A_j \xrightarrow{\quad} H_j$$

Figure 7: An infinite family of effective edge groupings

$G_j \times H_j$	$A_j \times H_j$	$H_j \times H_j$
$G_j \times A_j$	$A_j \times A_j$	$H_j \times A_j$
$G_j \times G_j$	$A_j \times G_j$	$H_j \times G_j$

Figure 8: An infinite family of effective square groupings

of finite effective groupings of $X - \Gamma_j(X)$ and taking $\Gamma_j \overset{\text{def}}{=} \pi_1(\Gamma_j(X), \sigma_0)$. All these square-complexes of groups will have the same universal cover $\Delta = \widetilde{\Gamma_j(X)} = T^1 \times T^2$ - a product of two trees.

First take an infinite increasing family of effective edges of groups as in figure 7. With $[H_j : A_j] = [G_j : A_j] = n$. An example for such an infinite family was first given by Djokovic in [Djo80] see also [BK90, section 7.14]. Using this we construct an infinite family of effective groupings of a square $G_j(S) = (S, G_{j,\sigma}, \psi_{j,a}, g_{a,b} = 0)$ just by taking a product as in figure 8. The universal cover of all these squares of groups will be the product of the universal cover of the corresponding edge of groups by itself that is $\Theta = T^n \times T^n$. Now construct $\Gamma_j(X) = (X, \Gamma_{j,\sigma}, \psi_{j,a}, \gamma_{a,b} = 0)$ by putting a copy of $G_j(S)$ on every 2-cell of X. We will prove

Theorem 7.1. *This construction gives an infinite strongly increasing series of irreducible lattices acting effectively on a product of two regular trees.*

Proof.
$\Gamma_j(X)$ is covered by a product of trees (See to figure 9).
Take a vertex $\sigma \in V(X)$. By theorem 2.2 it suffices to check that $Lk(\tilde{\sigma})$

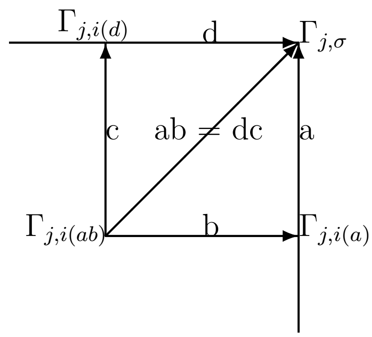

Figure 9: A typical vertex link in $G_j(S)$

is a full bipartite graph. The basic idea is simple, use first the link condition on X and then on $G_j(S)$ in order to establish a link condition on $\Gamma_j(X)$. In order to be precise we use the explicit construction of $Lk(\tilde{\sigma})$ given in section A.11 (see also [Hae91, Section 5.6]). Take two vertices $(\gamma\psi_{j,a}(\Gamma_{j,i(a)}), a), (\delta\psi_{j,d}(\Gamma_{j,i(d)}), d) \in Lk(\tilde{\sigma})$. where a, d are a vertical and horizontal edges in $Lk(\sigma)$ respectively. By the link condition on X there is a unique 1-cell in $Lk(\sigma)$, say $ab = dc$ connecting a and d. By the link condition on $G_j(S)$ there is a unique 1-cell in $Lk(\tilde{\sigma})$ connecting $(\gamma\psi_{j,a}(\Gamma_{j,i(a)}), a)$ and $(\delta\psi_{j,d}(\Gamma_{j,i(d)}), d)$ given by $(\gamma\psi_{j,d}(\Gamma_{j,i(d)}) \cap \delta\psi_{j,a}(\Gamma_{j,i(a)}), ab) = (\epsilon\psi_{j,ab}(\Gamma_{j,i(b)}), ab)$. This shows that $Lk(\tilde{\sigma})$ is a full bipartite graph and completes the proof.
Γ_j is an irreducible lattice.
We construct an immersion of complexes of groups $\Phi = \Phi_j = (id, 0, 0) : X \hookrightarrow \Gamma_j(X)$ by taking all homomorphisms and twisting elements to be trivial, and taking the topological map to be the identity. This immersion gives rise by theorem A.7 to the following immersion of actions.

$$\begin{tikzcd} \Lambda \arrow[r, "\delta\Phi_{\sigma_0}"] \arrow[d] & \Gamma_j \arrow[d] \\ \Delta' \arrow[r, "\widetilde{\delta\Phi}"] & \Delta \end{tikzcd}$$

Since everything commutes with the projection maps we get a similar diagram:

$$\begin{tikzcd} \Lambda^i \arrow[r, "\delta\Phi_{\sigma_0}"] \arrow[d] & \Gamma_j^i \arrow[d] \\ T^{i'} \arrow[r, "\widetilde{\delta\Phi}"] & T^i \end{tikzcd}$$

But the action $\Lambda^i \rightsquigarrow T^{i'}$ is non discreet because we choose $\Lambda < \mathrm{Aut}(\Delta')$ as an irreducible lattice. And this implies that $\Gamma_j \rightsquigarrow T^i$ can not be a discreet. The groupings $\Gamma_j(X)$ are effective.
Construct an immersion of complexes of groups $\Phi = \Phi_j : G_j(S) \hookrightarrow \Gamma_j(X)$ by choosing any 2-cell in X and viewing him as a sub-complex of full groups of $\Gamma_j(X)$. Again by theorem A.5 there is an injective action morphism:

$$\begin{array}{ccc} \pi_1(G_j(S), \sigma_0) & \xrightarrow{\delta\Phi_{\sigma_0}} & \pi_1(\Gamma_j(X), \sigma_0) \\ \downarrow & & \downarrow \\ \Theta & \xrightarrow{\widetilde{\delta\Phi}} & \Delta \end{array}$$

Assume that the grouping $\Gamma_j(X)$ is not effective. The kernel of the action would appear as a normal subgroup in every cell stabilizer $\Gamma_{j,\sigma}$. Taking such an element in one of the groups in the image of $G_j(S)$ and pulling back we obtain an element of $\pi_1(G_j(S), \sigma_0)$ acting trivially on Θ - contradicting the fact that $G_j(S)$ is, by construction, effective. This completes the proof of the theorem. $\square$

A Notations and Theorems from the Bass Serre Theory Used in this Paper

The purpose of this appendix is to summarize all the definitions and theorems pertaining to the theory of complexes of groups that are used in this paper. Most of what is written here is taken from [Hae91] and the only reason for bringing it here is in order to make these notes more self contained.

We will deal here with square complexes of groups because we are interested in products of two trees which are square complexes. In [Hae91] Haefliger treats mainly simplicial complexes, But the theory is actually the same. Some simplifications here are possible, however, due to the fact that we are treating complexes of dimension two.

The theory of square complexes of groups is a generalization of the Bass-Serre theory for graphs of groups to higher dimensions. The main idea is the following. Given an action (not necessarily free) of a group on a simply connected square complex $G \rightsquigarrow Y$ with orbit space $X \overset{\mathrm{def}}{=} G \backslash Y$ which is again a square complex. We encode the action by putting on

X a structure of a square complex of groups denoted $G(X)$. An action morphism $(G \rightsquigarrow Y) \to (G' \rightsquigarrow Y')$ is encoded by a morphism of complexes of groups $G(X) \to G'(X')$. A square complex of groups constructed from such an action is called developable.

Given a square complex go groups $G(X)$ we associate to it a fundamental group $\pi_1(G(X), \sigma_0)$ and a universal cover $\widetilde{G(X)}$ and there is a natural action of $G(X)$ on $\widetilde{G(X)}$. If $G(X)$ is developable constructed from an action $G \rightsquigarrow Y$ the fundamental group and universal cover will reconstruct G and Y. If $G(X)$ is obtained from a free action ($G(X)$ is called a complex of trivial groups in such a case) this whole theory is reduced to the regular theory of the fundamental group and universal cover.

A.1 Basic Definitions

Let X be a square complex. Define $V(X)$ and $E(X)$ to be the vertices and edges of its barycentric subdivision. We let the edges of the barycentric subdivision come with their natural orientation. For $a \in E(X)$ let $i(a), t(a) \in V(X)$ denote its initial and terminal points with respect to this orientation. Two edges $a, b \in E(X)$ are called composable if $t(b) = i(a)$ and in this case we denote by $ab \in E(X)$ the edge with $i(ab) = i(b)\ ;\ t(ab) = t(a)$.

A.2 Square Complexes of Groups

A complex of group structure on X is a 4-tuple $(X, G_\sigma, \psi_a, g_{a,b})$ where

- G associates to each $\sigma \in V(X)$ a group G_σ.
- ψ associates a monomorphism $G_{i(a)} \to G_{t(a)}$ to every edge $a \in E(X)$
- g associates an element $g_{a,b} \in G_{t(a)}$ to every pair of composable edges $a, b \in E(X)$. satisfying the following:

$$\mathrm{Ad}(g_{a,b}) \circ \psi_{ab} = \psi_a \circ \psi_b \tag{A.1}$$

A.3 Morphisms

Let $G'(X') = (X', G'_\sigma, \psi'_a, g'_{a,b})$ be another square complex of groups. A morphism of square complexes of groups $\Phi : G(X) \to G'(X')$ is a triple $\Phi = (f, \phi_\sigma, g'_a)$ where:

- $f : X \to X'$ is a square complex morphism.
- $\phi_\sigma : G_\sigma \to G'_{f(\sigma)}$ is a homomorphism of groups associated with every vertex $\sigma \in V(X)$.
- g' associates an element $g'_a \in G'_{f(t(a))}$ to every edge $a \in E(X)$. Such that

$$\mathrm{Ad}(g'_a) \circ \psi'_{f(a)} \circ \phi_{i(a)} = \phi_{t(a)} \circ \psi_a \quad \forall a \in E(X) \tag{A.2}$$

$$\phi_{t(a)}(g_{a,b})g'_{ab} = g'_a \psi'_{f(a)}(g'_b) g'_{f(a),f(b)} \quad \forall a, b \in E(X);\; t(b) = i(a) \tag{A.3}$$

We sometimes call such a morphism a morphism of $G(X)$ to $G'(X')$ defined over f.

A.4 Changing a Complex or a Morphism by a coboundary

If $\Phi : G(X) \to G'(X')$ is given as in section A.3. and given elements $\overline{g}_\sigma \in G'_{f(\sigma)}$. We can construct another morphism $\overline{\Phi} = (f, \overline{\phi}_\sigma, \overline{g}_a) : G(X) \to G'(X')$ over f by defining:

$$\overline{\phi}_\sigma \overset{\text{def}}{=} \mathrm{Ad}(\overline{g}_\sigma) \circ \phi_\sigma \tag{A.4}$$

$$\overline{g}_a \overset{\text{def}}{=} \overline{g_{t(a)}} g'_a \psi'_{f(a)}(\overline{g}^{-1}_{i(a)}) \tag{A.5}$$

We say that $\overline{\Phi}$ is deduced form Φ by the coboundary of $\{\overline{g_\sigma}\}_{\sigma \in V(X)}$.

Given a complex of groups structure $G(X)$ on X All structures deduced from $G(X)$ by coboundarys over the identity morphism are called equivalent to $G(X)$ up to a coboundary.

A.5 Developable Complexes

Let G be a group acting without inversion on a simply connected square complex Y. Let $X \overset{\text{def}}{=} G\backslash Y$ be the orbit space which is again a square complex and let $p : Y \to X$ be the projection. We associate with the action $G \rightsquigarrow Y$ a complex of groups structure on X in the following way.

We first choose liftings over p

$$\begin{array}{ll} \{\tilde{\sigma}\}_{\sigma \in V(X)} & \\ \{\tilde{a}\}_{a \in E(X)} & \quad \textit{such that } i(\tilde{a}) = \widetilde{i(a)} \end{array} \tag{A.6}$$

and group elements:

$$\{h_a \in G\}_{a \in E(X)} \quad such\ that\ h_a(t(\tilde{a})) = \widetilde{t(a)} \tag{A.7}$$

now we define the complex of group structure $G(X) = (X, G_\sigma, \psi_a, g_{a,b})$ **by:**

$$G_\sigma \stackrel{\text{def}}{=} \text{Stab}_G(\tilde{\sigma}) \tag{A.8}$$

$$\psi_a \stackrel{\text{def}}{=} \text{Ad}(h_a) \tag{A.9}$$

$$g_{a,b} \stackrel{\text{def}}{=} h_a h_b h_{ab}^{-1} \tag{A.10}$$

It turns out that making a different choice in equation A.6 and in equation A.7, changes the complex of group structure on X **by a coboundary. A complex of groups constructed in such a way is called developable.**

A.6 Developable Morphisms

Given a morphism of actions $(\phi, \tilde{f}) : (G, Y) \rightarrow (G', Y')$ **We construct the corresponding developable complexes of groups** $G(X)$ **and** $G'(X')$ **on the orbit spaces** X **and** X' **as in section A.5 (using the same notation).** $\tilde{f}$ **induces a morphism on the orbit spaces** $f : X \rightarrow X'$**. We construct a morphism over** f **of the corresponding developable complexes of groups** $\Phi = (f, \phi_\sigma, g'_a) : G(X) \rightarrow G'(X')$**. Again we have to make some arbitrary choices. We select group elements**

$$\{k_\sigma \in G'\}_{\sigma \in V(X)} \quad such\ that\ k_\sigma(\tilde{f}(\tilde{\sigma})) = \widetilde{f(\sigma)} \tag{A.11}$$

And now we define

$$\phi_\sigma \stackrel{\text{def}}{=} \text{Ad}(k_\sigma) \circ \phi \tag{A.12}$$

$$g'_a \stackrel{\text{def}}{=} k_{t(a)} \phi(h_a) k_{i(a)}^{-1} (h'_{f(a)})^{-1} \tag{A.13}$$

One checks that this is a well defined and that changing the choices in equation A.11 changes Φ **by a coboundary.**

A.7 The Path Group

Given a complex of groups $G(X)$ **as in section A.2 we define its path group** $FG(X)$ **by generators and relations. The generators being:**

- all the elements of the groups $\cup_{\sigma \in V(X)} G_\sigma$
- all oriented edges of X denoted $E^{\pm} \stackrel{\text{def}}{=} \left\{a^+\right\}_{a \in E(X)}\big) \cup \big(\left\{a^-\right\}_{a \in E(X)}\big)$

And the relations:

- all the relations of the groups $\{G_\sigma\}_{\sigma \in V(X)}$.
- $(a^+)^{-1} = a^-$ and $(a^-)^{-1} = a^+$.
- $\psi_a(g) = a^- g a^+$ for $g \in G_{i(a)}$.
- $(ab)^+ = b^+ a^+ g_{a,b}$ for a, b a pair of composable edges in $E(X)$.

A.8 Reconstructing an Action From a Developable Complex of Groups

Theorem A.1. *A given square complex of groups is developable (i.e. obtained from an action $G \rightsquigarrow Y$) if and only if the natural homomorphisms $\{G_\sigma \to FG(X)\}_{\sigma \in V(X)}$ are all embedings . In this case we associate to $G(X)$ a group action on a simply connected square complex denoted $\pi_1(G(X), \sigma_0) \rightsquigarrow \widetilde{G(X)}$ reconstructing the original action $G \rightsquigarrow Y$.*

Definition A.2. *The group and square complex thus obtained are called the fundamental group and the universal cover of $G(X)$.*

In sections A.9 and A.10 bellow we describe the construction of the fundamental group and the universal cover of a given complex of groups.

A.9 The Fundamental Group

We give here two equivalent definitions of the fundamental group of a complex.

definition I Let T be a maximal spanning tree in the one skeleton of the barycentric subdivision of X. Let N be the normal subgroup generated by all edges of T $N = \langle a^+ : a \in E(T) \rangle$. Now define

$$\pi_1(G(X), T) \stackrel{\text{def}}{=} \frac{FG(X)}{N} \tag{A.14}$$

definition II **Choose a vertex $\sigma_0 \in V(X)$. For every vertex $\sigma \in V(X)$ a path from σ_0 to σ is a sequence $(g_0, e_1, g_1, e_2, \dots e_n, g_n)$ with $e_j \in E^{\pm}(X)$, $t(e_i) = i(e_{i+1})$, $i(e_1) = \sigma_0$, $t(e_n) = \sigma$, and $g_j \in G_{t(e_j)} = G_{i(e_j)}$. To every such path corresponds an element of $FG(X)$ in the obvious way: $(g_0, e_1, g_1, e_2, \dots e_n, g_n) \to g_0 e_1 g_1 e_2 \centerdot \dots \centerdot e_n g_n$. Two paths are said to be homotopic if they represent the same element of $FG(X)$. We denote the image in $FG(X)$ of all paths from σ_0 to σ under this correspondence by $\pi_1(G(X), \sigma_0, \sigma)$ and define the fundamental group**

$$\pi_1(G(X), \sigma_0) \stackrel{\text{def}}{=} \pi_1(G(X), \sigma_0, \sigma_0) \tag{A.15}$$

We may think of the fundamental group $\pi_1(G(X), \sigma_0)$ as the group of all loops from σ_0 to itself divided by the equivalence relation of homotopy and with the group action of concatenation of paths.

Theorem A.3. *Given a complex of groups $G(X)$, a vertex $\sigma_0 \in V(X)$ and a maximal spanning tree - T in the one skeleton of the barycentric subdivision of X. There are obvious homomorphisms:*

$$\pi_1(G(X), \sigma_0) \to FG(X) \to \pi_1(G(X), T) \tag{A.16}$$

and the composition of the two homomorphism induces an isomorphism

$$\pi_1(G(X), \sigma_0) \cong \pi_1(G(X), T) \tag{A.17}$$

A.10 The Universal Cover

Assume a complex of groups $G(X)$ is given such that all the homomorphisms $\{G_\sigma \to FG(X)\}_{\sigma \in V(X)}$ are embedings, so that $G(X)$ is developable by theorem A.1. In this section we describe the construction of the universal cover $\widetilde{G(X)}$.

choose a maximal spanning tree T in the one skeleton of the barycentric subdivision of X The vertices of the barycentric subdivision of $\widetilde{G(X)}$ are given by:

$$\begin{aligned} V(\widetilde{G(X)}) &\stackrel{\text{def}}{=} \bigcup_{\sigma \in V(X)} \frac{\pi_1(G(X), T)}{G_\sigma} \\ &= \{(gG_\sigma, \sigma) : \sigma \in V(X), g \in \pi_1(G(X), T)\} \end{aligned} \tag{A.18}$$

The edges of the barycentric subdivision of $\widetilde{G(X)}$ are given by:

$$\begin{aligned} E(\widetilde{G(X)}) &\stackrel{\text{def}}{=} \bigcup_{a \in E(X)} \frac{\pi_1(G(X), T)}{G_{i(a)}} \\ &= \{(gG_{i(a)}, a) : a \in E(X), g \in \pi_1(G(X), T)\} \end{aligned} \tag{A.19}$$

We define the initial and terminal vertices of an edge by:

$$i(gG_{i(a)}, a) = (gG_{i(a)}, i(a)) \tag{A.20}$$

$$t(gG_{i(a)}, a) = (ga^+G_{t(a)}, t(a)) \tag{A.21}$$

We define the natural projection $p : G(X) \to X$ by:

$$p((gG_\sigma, \sigma)) = \sigma \tag{A.22}$$

$$p((gG_{i(a)}, a)) = a \tag{A.23}$$

Finely we define the action of $\pi_1(G(X), T) \rightsquigarrow \widetilde{G(X)}$ by the formulas:

$$h(gG_\sigma, \sigma) = (hgG_\sigma, \sigma) \tag{A.24}$$

$$h(gG_{i(a)}, a) = (hgG_{i(a)}, a) \tag{A.25}$$

One defines the complex $\widetilde{G(X)}$ as the geometric realization of the category associated with this graph. It is easy to check that everything here is well defined.

A.11 Links: Reconstructing Local Information About the Universal Cover

It is possible to extract local data about the universal cover of a given developable complex $G(X) = (X, G_\sigma, \psi_a, g_{a,b})$ from the local data of about complex itself. Explicitly we show that we can construct the link of a vertex $(gG_\sigma, \sigma) \in \widetilde{G(X)}$ only from the information given in $G(X)$ about the star of σ. In particular this link is independent of the choice of lifting (i.e. of the choice of g) we therefore define:

Definition A.4. *Given a developable group complex $G(X)$ and a vertex $\sigma \in V(X)$ we define the link of σ in the complex of groups to be the link of any lifting of σ to the universal cover. We denote this link by $\mathrm{Lk}(\tilde{\sigma}) = \mathrm{Lk}_{g(X)}(\tilde{\sigma})$. When the complex $G(X)$ is understood the subscript $G(X)$ will be omitted.*

remark this should be distinguished from $\mathrm{Lk}(\sigma) = \mathrm{Lk}_X(\sigma)$ which is the link of σ in X.

We define the vertices of the barycentric subdivision of $\mathrm{Lk}(\tilde{\sigma})$ by:

$$\begin{aligned} V(\mathrm{Lk}(\tilde{\sigma})) &\stackrel{\text{def}}{=} \bigcup_{a \in E(X); t(a)=\sigma} \frac{G_\sigma}{\psi_a(G_{i(a)})} \\ &= \{(g\psi_a(G_{i(a)}), a) : t(a) = \sigma; g \in G_\sigma\} \end{aligned} \tag{A.26}$$

We define edges of the barycentric subdivision of $\mathrm{Lk}(\tilde{\sigma})$ by:

$$\begin{aligned} E(\mathrm{Lk}(\tilde{\sigma})) &\stackrel{\text{def}}{=} \bigcup_{a,b \in E(X); t(a)=\sigma; t(b)=i(a)} \frac{G_\sigma}{\psi_{ab}(G_{i(b)})} \\ &= \{(g\psi_{ab}(G_{i(b)}), a, b) : t(a) = \sigma; t(b) = i(a); g \in G_\sigma\} \end{aligned} \tag{A.27}$$

The initial and terminal vertices of an edge are defined by:

$$i(g\psi_{ab}(G_{i(b)}), a, b) = (g\psi_{ab}(G_{i(b)}), ab) \tag{A.28}$$

$$t(g\psi_{ab}(G_{i(b)}), a, b) = (gg_{a,b}^{-1}\psi_a(G_{i(a)}), a) \tag{A.29}$$

One checks that when $G(X)$ is developable this indeed gives the link of a lifting of σ to the universal cover.

A.12 Reconstructing a Morphism of Actions

We recall that in [Bas93, propositions 2.4,2.7;section 2.9] to every morphism of graphs of groups $\Phi = (f, \phi_\sigma, g'_a) : G(X) \to G(X')$ we can associate an action morphism of actions:

$$\left(\delta\Phi_{\sigma_0}, \widetilde{\delta\Phi}\right) : \left(\pi_1(G(X), \sigma_0), (\widetilde{G(X), \sigma_0}\right) \to \left(\pi_1(G(X'), f(\sigma_0), (\widetilde{G(X'), f}(\sigma_0))\right) \tag{A.30}$$

Local conditions on Φ are given there to determine when is the morphism an immersion (resp: a covering) in which case $\delta\Phi_{\sigma_0}$ and $\widetilde{\delta\Phi}$ are immersions (resp: $\delta\Phi_{\sigma_0}$ is an immersion of groups and $\widetilde{\delta\Phi}$ is an isomorphism of trees). A similar theorem holds for complexes of groups of two (or higher) dimensions whose universal covers are products of trees. For arbitrary complexes of groups of higher dimension only a partial generalization of the theorem holds.

It should be noticed that a morphism Φ of graphs of groups, as Haefliger defines it in [Hae91, section 2.2] (which is also the definition used by us in section A.3), carries actually less information then the morphisms defined by Bass [Bas93, section 2.1]. For this reason it is not possible to reconstruct the morphisms Φ_{σ_0} and $\tilde{\Phi}$ as defined in [Bas93, Proposition 2.4] but only the morphisms $\delta\Phi_{\sigma_0}$ and $\widetilde{\delta\Phi}$ defined in [Bas93, section 2.9].

Theorem A.5. *Let $G(X) = (X, G_\sigma, \psi_a, g_{a,b})$ and $G(X') = (X', G'_\sigma, \psi'_a, g'_{a,b})$ be two developable complexes of groups and let $\Phi = (f, \phi_a, g'_a) : G(X) \to G(X')$ be a morphism of complexes of groups. Then Φ induces a morphism of actions*

$$\begin{array}{ccc} \pi_1(G(X),\sigma_0) & \xrightarrow{\delta\Phi_{\sigma_0}} & \pi_1(G'(X'), f(\sigma_0)) \\ \downarrow & & \downarrow \\ \widetilde{G(X)} & \xrightarrow{\widetilde{\delta\Phi}} & \widetilde{G'(X')} \end{array}$$

where $\delta\Phi_{\sigma_0}$ is defined on generators of $\pi_1(G(X),\sigma_0)$ by:

$$\begin{aligned} \delta\Phi_{\sigma_0}(g) &= \phi_\sigma(g) \quad \forall\sigma \in V(X)\ g \in G_\sigma \\ \delta\Phi_{\sigma_0}(a^+) &= f(a)^+(g'_a)^{-1} \quad \forall a \in E(X) \end{aligned} \tag{A.31}$$

and $\widetilde{\delta\Phi}$ is defined on the universal cover by:

$$\widetilde{\delta\Phi}(gG_\sigma, \sigma) = (\delta\Phi_{\sigma_0}(g)G_{f(\sigma)}, f(\sigma)) \quad \forall\sigma \in V(X)\ g \in \pi_1(G(X),\sigma_0) \tag{A.32}$$

$$\widetilde{\delta\Phi}(gG_{i(a)}, a) = (\delta\Phi_{\sigma_0}(g)G_{i(f(a))}, f(a)) \quad \forall a \in E(X)\ g \in \pi_1(G(X),\sigma_0) \tag{A.33}$$

Proof: We first define a homomorphism of the corresponding path groups (see section A.7): $\Phi : FG(X) \to FG(X')$. We define Φ on the generators of $FG(X)$ as follows:

$$\Phi(g) = \phi_\sigma(g) \qquad \forall\sigma \in V(X)\ g \in G_\sigma \tag{A.34}$$

$$\Phi(a^+) = f(a)^+(g'_a)^{-1} \qquad \forall a \in E(X) \tag{A.35}$$

We have to verify the defining relations of $FG(X)$ are not in contradiction with the definition.

$$\begin{aligned} \Phi(a^- g a^+) &= \Phi(a^-)\Phi(g)\Phi(a^+) \\ &= g'_a f(a)^- \phi_{i(a)}(g) f(a)^+ (g'_a)^{-1} \\ &= g'_a \psi'_{f(a)}(\phi_{i(a)}(g))(g'_a)^{-1} \\ &= \mathrm{Ad}(g'_a) \circ \psi'_{f(a)} \circ \phi_{i(a)}(g) \\ &= \phi_{t(a)} \circ \psi_a(g) \\ &= \Phi(\psi_a(g)) \end{aligned} \tag{A.36}$$

$$
\begin{aligned}
\Phi((ab)^{+}) &= f(ab)^{+}(g'_{ab})^{-1} \\
&= f(b)^{+}f(a)^{+}g'_{f(a),f(b)}(g'_{ab})^{-1} \\
&= f(b)^{+}f(a)^{+}\psi'_{f(a)}(g'_{b})^{-1}\psi'_{f(a)}(g'_{b})g'_{f(a),f(b)}(g'_{ab})^{-1} \\
&= f(b)^{+}f(a)^{+}f(a)^{-}(g'_{b})^{-1}f(a)^{+}\psi'_{f(a)}(g'_{b})g'_{f(a),f(b)}(g'_{ab})^{-1} \\
&= f(b)^{+}(g'_{b})^{-1}f(a)^{+}\psi'_{f(a)}(g'_{b})g'_{f(a),f(b)}(g'_{ab})^{-1} \\
&= f(b)^{+}(g'_{b})^{-1}f(a)^{+}(g'_{a})^{-1}g'_{a}\psi'_{f(a)}(g'_{b})g'_{f(a),f(b)}(g'_{ab})^{-1} \\
&= f(b)^{+}(g'_{b})^{-1}f(a)^{+}(g'_{a})^{-1}\phi_{t(a)}(g_{a,b}) \\
&= \Phi(b^{+})\Phi(a^{+})\Phi(g_{a,b}) \\
&= \Phi(b^{+}a^{+}g_{a,b})
\end{aligned}
\tag{A.37}
$$

This shows that $\Phi : FG(X) \to FG(X')$ is well defined. furthermore we can see that Φ maps $\pi_1(G(X), \sigma_0)$ into $\pi_1(G'(X'), \sigma_0)$ because if $c = (g_0 e_1 g_1 e_2 \ldots e_n g_n)$ is and edge path in the barycentric subdivision of X, with $i(e_1) = t(e_n) = \sigma_0$ i.e. $c \in \pi_1(G(X), \sigma_0)$ then

$$
\Phi(c) = \big(\phi_{\sigma_0}(g_0) f(e_1)^{+} \left(g'_{e_1}\phi_{t(e_1)}(g_1)\right) f(e_2)^{+} \ \ldots \ \phi_{\sigma_0}(g_n)\big) \tag{A.38}
$$

Thus we define $\delta\Phi_{\sigma_0}$ by

$$
\delta\Phi_{\sigma_0} \stackrel{\text{def}}{=} \Phi|_{\pi_1(G(X),\sigma_0)} \tag{A.39}
$$

which is compatible with the definition in equation A.31 in theorem A.5. We also define as in equation A.32 in theorem A.5 the morphism of the universal covers $\widetilde{\delta\Phi} : \widetilde{G}(X) \to G(X')$.

$$
\widetilde{\delta\Phi}(gG_{\sigma}, \sigma) = (\delta\Phi_{\sigma_0}(g)G_{f(\sigma)}, f(\sigma)) \quad \forall \sigma \in V(X)\ g \in \pi_1(G(X), \sigma_0) \tag{A.40}
$$

$$
\widetilde{\delta\Phi}(gG_{i(a)}, a) = (\delta\Phi_{\sigma_0}(g)G_{i(f(a))}, f(a)) \quad \forall a \in E(X)\ g \in \pi_1(G(X), \sigma_0) \tag{A.41}
$$

Again we have to check a few things. first we check that $\widetilde{\delta\Phi}$ commutes with i and t which means that $\widetilde{\delta\Phi}$ is a well defined as graph morphism on the one skeleton of the barycentric subdivisions of $\widetilde{G(X)}$ and $\widetilde{G(X')}$. Since the universal cover can be constructed as the complex which is the nerve of the category associated with this directed graph we conclude that $\widetilde{\delta\Phi}$ is

a well defined category of square complexes.

$$\begin{aligned} i(\widetilde{\delta\Phi}(gG_{i(a)}, a)) &= i(\delta\Phi_{\sigma_0}(g)G_{i(f(a))}, f(a)) \\ &= (\delta\Phi_{\sigma_0}(g)G_{f(i(a))}, f(i(a))) \\ &= \widetilde{\delta\Phi}(gG_{i(a)}, i(a)) \\ &= \widetilde{\delta\Phi}(i(gG_{i(a)}, a)) \end{aligned} \tag{A.42}$$

$$\begin{aligned} t(\widetilde{\delta\Phi}(gG_{i(a)}, a)) &= t(\delta\Phi_{\sigma_0}(g)G_{i(f(a))}, f(a)) \\ &= (\delta\Phi_{\sigma_0}(g)f(a)^+G_{t(f(a))}, t(f(a))) \\ &= (\delta\Phi_{\sigma_0}(g)f(a)^+(g'_a)^{-1}g'_aG_{t(f(a))}, t(f(a))) \\ &= (\delta\Phi_{\sigma_0}(ga^+)G_{f(t(a))}, f(t(a))) \\ &= \widetilde{\delta\Phi}(ga^+G_{t(a)}, t(a)) \\ &= \widetilde{\delta\Phi}(t(gG_{i(a)}, a)) \end{aligned} \tag{A.43}$$

It remains to show that $\widetilde{\delta\Phi}$ is $\delta\Phi_{\sigma_0}$ equivariant.

$$\begin{aligned} \delta\Phi_{\sigma_0}(g)(\widetilde{\delta\Phi}(hG_\sigma, \sigma)) &= \delta\Phi_{\sigma_0}(g)(\delta\Phi_{\sigma_0}(h)G_{f(\sigma)}, f(\sigma)) \\ &= (\delta\Phi_{\sigma_0}(g)\delta\Phi_{\sigma_0}(h)G_{f(\sigma)}, f(\sigma)) \\ &= (\delta\Phi_{\sigma_0}(gh)G_{f(\sigma)}, f(\sigma)) \\ &= \widetilde{\delta\Phi}(ghG_\sigma, \sigma) \\ &= \widetilde{\delta\Phi}(g(hG_\sigma, \sigma)) \end{aligned} \tag{A.44}$$

A similar calculation holds for the edges of the barycentric subdivision of $\widetilde{G(X)}$. This completes the proof of theorem A.5 $\square$.

A.13 Deducing Global Information from Local Information About the Action Morphism

Definition A.6. Φ *is called an immersion (resp a covering) if for every* $\sigma \in V(X)$ *:*

- ϕ_σ *is injective.*
- *the morphism induced by* $\widetilde{\delta\Phi}$ *on the links* $Lk(\tilde{\sigma}) \to Lk(\widetilde{f(\sigma)})$ *is injective (resp a isomorphism).*

Theorem A.7. *Using the notation of theorem A.5*

1. *Φ is a covering iff $\delta\Phi_{\sigma_0}$ is injective and $\widetilde{\delta\Phi}$ is an isomorphism.*

2. *If we assume that $\widetilde{G(X)}$ and $\widetilde{G(X')}$ are products of trees. Φ is a covering if $\delta\Phi_{\sigma_0}$ is injective and $\widetilde{\delta\Phi}$ is injective.*

Proof: The proof follows almost verbally the proof in [Bas93, 2.7] First observe that for $\widetilde{\delta\Phi}$ to be an isomorphism is equivalent to being locally an isomorphism because if $\widetilde{\delta\Phi}$ is a locally isomorphism it must be a covering (because it is a map of square complexes the fiber must be discreet). But $\widetilde{G(X')}$ is simply connected and the theorem follows.

If we assume $\widetilde{G(X)}$ and $\widetilde{G(X')}$ are products of trees injectivity is equivalent to local injectivity. This definitely holds for trees, and a morphism of a product of trees $Z : T^1 \times T^2 \to Y^1 \times Y^2$ must split as a product of two morphisms $Z = Z^1 \times Z^2$.

Now assume $\widetilde{\delta\Phi}$ is injective (resp: bijective) and locally injective (resp: locally bijective) and we want to prove that:
$\delta\Phi_{\sigma_0}$ is injective $\Leftrightarrow$ ϕ_{sigma} is injective $\quad \forall\sigma \in V(X)$

We first prove ($\Leftarrow$) take $g \in \ker(\delta\Phi_{\sigma_0})$. Since $\widetilde{\delta\Phi}$ is injective and $\delta\Phi_{\sigma_0}$ equivariant g must act trivially in $\widetilde{G(X)}$ especially $g(\sigma_0) = \sigma_0$ so that $g \in G_{\sigma_0}$. Now we can write explicitly the expression $\delta\Phi_{\sigma_0}(g) = \phi_{\sigma_0}(g) = 1$. But by assumption ϕ_{σ_0} is injective so $g = 0$.

Now ($\Rightarrow$) is obvious by the definition of $\delta\Phi_{\sigma_0}$. $\square$

A.14 A Counter Example

It is easy to construct examples showing that the requirement that $\widetilde{G(X)}$ and $\widetilde{G(X')}$ be products of trees. In theorem A.7 part 2 is indeed necessary. Such an example (even on square complexes with trivial groups) is demonstrated in the figure 10.

Figure 10: An immersion of square complexes need not induce an injection on universal covers: An immersion of a cylinder into a cone

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