

TOWARD THE ABELIAN ROOT GROUPS CONJECTURE FOR SPECIAL MOUFANG SETS

YOAV SEGEV*

ABSTRACT. We prove that if a root group of a special Moufang set contains an element of order $p \equiv 1 \pmod{4}$ then it is abelian.

1. INTRODUCTION

Moufang sets are the Moufang buildings of rank one. They were introduced by J. Tits [T] as a tool for studying algebraic groups of relative rank one. They are essentially equivalent to split BN -pairs of rank one, and to Timmesfeld's abstract rank one groups (see [Ti]).

Let us recall that a Moufang set is a doubly transitive permutation group such that the point stabilizer contains a normal subgroup which is regular on the remaining points. These regular normal subgroups are called the *root groups* and they are assumed to be conjugate and to generate the whole group. In [DW, DS1, DS2] the notation $\mathbb{M}(U, \tau)$ is used for a Moufang set (and this notation is of course explained there). The group U in this notation is isomorphic to any one of the root groups of the Moufang set.

Recall that $\mathbb{M}(U, \tau)$ is *special* iff, $(-a)\tau = -(a\tau)$, for all $a \in U^*$. This paper makes a contribution to the following conjecture.

Abelian root groups conjecture. *Let $\mathbb{M}(U, \tau)$ be a special Moufang set. Then U is abelian.*

Thus the conjecture asserts that special Moufang sets have abelian root groups. In [S2] we proved the converse: that proper (i.e. not sharply 2-transitive) Moufang sets with abelian root groups are special. We prove

Theorem A. *Let $\mathbb{M}(U, \tau)$ be a special Moufang set. Suppose there exists $a \in U^*$ whose order is $p \equiv 1 \pmod{4}$. Then U is abelian.*

In [DST, §6] it is shown that if U^* contains involutions (i.e. elements of order 2), then U is a group of exponent 2 (and hence abelian). Thus since $a \in U^*$ has order $p \equiv 1 \pmod{4}$, [DS1, Proposition 4.6] implies that U is a uniquely 2-divisible group. Section 2 deals with such groups, in fact what we need is information on regular automorphisms of order 2 and 4 of such

Date: July 23, 2008.

*Partially supported by BSF grant no. 2004-083.

groups. Several of the results of §2 are taken from [K], but we add a little more.

2. REGULAR AUTOMORPHISMS OF ORDER 4 ACTING ON UNIQUELY 2-DIVISIBLE GROUPS

In this section U is a uniquely 2-divisible group. This means that each element in U has a unique square root. We use additive notation for U although *we do not assume that U is abelian*. The unique square root of an element $x \in U^*$ is thus denoted $x \cdot \frac{1}{2}$. Also, for $n \in \mathbb{Z}$ we write $x \cdot n$ for the n -th power of x . We will use the usual notation $x^y = -y + x + y$ and $[x, y] = -x - y + x + y$. Notice however that x^{-y} denotes $y + x - y$ and not $-y - x + y = (-x)^y$. Few more words of caution. Notice that 0 is the neutral element of U , so for $x, y \in U^*$, $[x, y] = 0$ indicates that x and y commute, and that the conjugation $u^0 = u$, for $u \in U$ (of course $u \cdot 0 = 0$).

We let $\eta \in \text{Aut}(U)$ be a regular automorphism of order 4. Regular means that the implication $y\eta = y \implies y = 0$ holds for all $y \in U$ (i.e. η is a fixed point free automorphism). For any automorphism ν of U of order 2 we let

$$T_\nu := C_U(\nu) \quad \text{and} \quad S_\nu := \{x \in U \mid x\nu = -x\}.$$

and we let

$$T := T_{\eta^2} \quad \text{and} \quad S := S_{\eta^2}.$$

The following lemma comes from [K, (3.3) and (3.4), p. 281].

Lemma 2.1. *Let $\nu \in \text{Aut}(U)$ have order 2. Then*

- (1) *each element $x \in U$ can be written uniquely as a sum $x = t + s$, with $t \in T_\nu$ and $s \in S_\nu$; in particular,*
- (2) *each element of U can be written uniquely as a sum $x = t + s$, with $t \in T$ and $s \in S$.*
- (3) *if ν is regular, then ν inverts U so U is abelian, thus*
- (4) *T is abelian and η inverts T (i.e. $t\eta = -t$, for all $t \in T$).*

Proof. (1): Let $x \in U^*$ and suppose $x = t + s$, with $t \in T_\nu$ and $s \in S_\nu$. Then

$$x + (-x + x\nu) \cdot \frac{1}{2} = (t + s) + (-s - t + t - s) \cdot \frac{1}{2} = t + s - s = t.$$

so t is uniquely determined by x and then $s = -t + x$ is also uniquely determined by x .

Let now $x \in U^*$ be arbitrary. Then

$$\begin{aligned} (x + (-x + x\nu) \cdot \frac{1}{2})\nu &= x\nu + (-x\nu + x) \cdot \frac{1}{2} = x - (-x\nu + x) + (-x\nu + x) \cdot \frac{1}{2} \\ &= x - (-x\nu + x) \cdot \frac{1}{2} = x + (-x + x\nu) \cdot \frac{1}{2}. \end{aligned}$$

Hence $t := x + (-x + x\nu) \cdot \frac{1}{2} \in T$ and $s = -t + x = (-x\nu + x) \cdot \frac{1}{2} \in S$.

(2): This follows from (1) applied to η^2 .

(3): Since ν is regular, this means that $T_\nu = 0$, so by (1), $U = S_\nu$, and (3) holds.

(4): Note that T is uniquely 2-divisible, and that the restriction of η to T is a regular automorphism of T of order 2, so (4) follows from (3). $\square$

Lemma 2.2 (see (3.5) in [K], p. 281). (1) *If $\nu \in \text{Aut}(U)$ has order 2, then $s^t \in S_\nu$, for all $t \in T_\nu$ and $s \in S_\nu$;*
 (2) *$\langle S \rangle$ is a normal subgroup of U and $U/\langle S \rangle$ is abelian.*

Proof. (1): Let $t \in T_\nu$ and $s \in S_\nu$, then $(s^t)\nu = (s\nu)^{t\nu} = (-s)^t = -(s^t)$, and (1) holds.

(2): Set $N := \langle S \rangle$. By Lemma 2.1(2), $U = NT$. By (1), T normalizes N , so $N \trianglelefteq U$. Then, since by Lemma 2.1(4) T is abelian, U/N is abelian. $\square$

The following proposition generalizes some of the results in [K, (5.1), p. 287].

Proposition 2.3. (1) *Let ν be any regular automorphism of U . Suppose $x, y \in U$ satisfy: $x\nu = x^{(-y+y\nu)}$. Then $x = 0$.*
 (2) *Let $y \in U^*$. Using Lemma 2.1(2), write $-y + y\eta = -t - s$, with $t \in T$ and $s \in S$. Then $[s^t, s\eta] = 0$.*
 (3) *Let $s \in S$. Then $[s, s\eta] = 0$.*
 (4) *Let $t \in T$ and $s \in S$. Then $[s^t, s\eta] = 0$.*

Proof. (1): We have $(x^{-y})\nu = x^{-y}$, so since ν is regular it follows that $x^{-y} = 0$ and then also $x = 0$.

(2): By Lemma 2.1(4) we have

$$[s^t, s\eta]\eta = [(s\eta)^{-t}, -s] = [s, (s\eta)^{-t}]^{-s} = [s^t, s\eta]^{-t-s} = [s^t, s\eta]^{(-y+y\eta)},$$

so (2) follows from (1) with η in place of ν and $[s^t, s\eta]$ in place of x .

(3): Set $x := -s + s\eta$, and write $x = -t - u$ with $t \in T$ and $u \in S$. By (2), $[u^t, u\eta] = 0$. Now $x + x\eta = -t - u + t - (u\eta)$, so $x + x\eta$ commutes with $u\eta$. But $x + x\eta = (-s + s\eta) + (-s\eta) - s = -s \cdot 2$. Hence $u\eta$ commutes with $s \cdot 2$, and since U is uniquely 2-divisible, $u\eta$ commutes with s . Thus $-u = (u\eta)\eta$ commutes with $s\eta$, and we see that u commutes with $s\eta$.

Now $-t = x + u = -s + (s\eta + u)$. Notice that since $s\eta$ and u commute, $s\eta + u \in S$. Set $v := s\eta + u$. Then $-t = -s + v$ and hence

$$-s + v = -t = (-t)\eta^2 = s - v.$$

It follows that $-s \cdot 2 = -v \cdot 2$, so by unique 2-divisibility, $s = v$ and $t = 0$. It follows that $-s + s\eta = -u$ and since $[u, s\eta] = 0$ we see that $[s, s\eta] = 0$.

(4): Let $t \in T$ and $s \in S$. By Lemma 2.2(1), $s^t \in S$. By (3), $(s^t)\eta$ commutes with s^t . But $(s^t)\eta = (s\eta)^{-t}$, so we see that $[s^{t \cdot 2}, s\eta] = 0$, replacing t with $t \cdot \frac{1}{2}$, we get (4). $\square$

3. THE PROOF OF THEOREM A

Throughout this section p is a prime such that $p \equiv 1 \pmod{4}$. We let $\mathbb{M}(U, \tau)$ be a special Moufang set and we assume that $a \in U^*$ is such that $|a| = p$. By [DST, §6] and [DS1, Proposition 4.6], U is a uniquely 2-divisible group. Our goal is to show that U is abelian. So assume toward contradiction that U is not abelian.

We now briefly recall several facts about special Moufang sets. We always denote $X := U \cup \{\infty\}$ and by G the little projective group of $\mathbb{M}(U, \tau)$. The letter H is reserved for $H := G_{0, \infty}$, and recall that $H \leq \text{Aut}(U)$. Recall from [DS1, Proposition 3.9] that

$$\mu_x^h = \mu_{xh} \text{ and } \mu_x^{\mu_y} = \mu_{-x\mu_y}, \quad \forall x, y \in U^* \text{ and } h \in H.$$

Here are some useful properties of U . By [DS1, Proposition 4.6] for every element $x \in U^*$ either $|x| = q$, where q is an odd prime or $|x| = \infty$, where $|x|$ is the order of x . Furthermore, by [DST, Proposition 5.3], $C_U(x)$ is a group of exponent q in the first case and $C_U(x)$ is a uniquely divisible torsion free group in the second case. In particular, if $C_U(x)$ is abelian, then $C_U(x)$ is a vector space over $\mathbb{F}_q$ or $\mathbb{Q}$ respectively.

We use the following observation which follows from e.g. [DS1, Proposition 4.10]:

$$(3.1) \quad \mu_x^2 = \mu_{x \cdot m}^2, \quad \forall x \in U^* \text{ with } |x| < \infty \text{ and all } 1 \leq m < |x|.$$

Indeed let $|x| = q$ and in [DS1, Proposition 4.10(3)] take k to be a generator of the multiplicative group $\mathbb{F}_q^*$. Then k is a nonsquare in $\mathbb{F}_q^*$. Now in the notation of [DS1, Proposition 4.10(3)], take ℓ such that $k^\ell = -1$ (in $\mathbb{F}_q^*$) and take $\ell' = 0$, so that $N = -1$. Then Parts (3) and (5) of [DS1, Proposition 4.10] show that $\mu_a^2 = \mu_{-a}^2 = \mu_{a \cdot k}^2$ and [DS1, Proposition 4.10(4)] shows that $\mu_{a \cdot t^2} = \mu_a^2$ and $\mu_{a \cdot kt^2} = \mu_{a \cdot k}^2$, for all $t \in \mathbb{F}_q^*$, so (3.1) holds. Finally see [S1, §3] for the notion of a root subgroup.

Notation 3.1. Let $x \in U^*$. We denote

- (1) $\sqrt{-1}$ is an element of the field $\mathbb{F}_p = \{0, \dots, p-1\}$ whose square is -1 .
- (2) $T_x := C_U(\mu_x^2)$.
- (3) $S_x := \{y \in U^* \mid y\mu_x^2 = -y\}$.
- (4) if $|x| = p$ we denote $\eta_x = \mu_x \mu_{x \cdot \sqrt{-1}}$.

We also let

$$T := T_a, \quad S := S_a \quad \text{and} \quad \eta = \eta_a.$$

Lemma 3.2. *Let $x \in U^*$ with $|x| = p$. Then*

- (1) *if $\mu_x^2 = 1$ then U is abelian, so μ_x has order 4;*
- (2) *T_x is a root subgroup of U and T_x is an elementary abelian (possibly infinite) p -subgroup of U .*

Proof. (1): Note that given distinct $y, z \in U^*$, the permutation action of $G_{y,z}$ on $X \setminus \{z\}$ is equivalent to the action of $G_{y,z}$ on U_y via conjugation.

By [DST, Proposition 7.7(4)] the only fixed points of μ_x on $X = U \cup \{\infty\}$ are $x \cdot \sqrt{-1}$, $-x \cdot \sqrt{-1}$, and thus applying the above on $y = x \cdot \sqrt{-1}$, $z = -x \cdot \sqrt{-1}$ we see that $C_{U_y}(\mu_x) = 1$. Now U_y is a uniquely 2-divisible group. Hence if $\mu_x^2 = 1$, then, by Lemma 2.1(3) (with U_y in place of U and conjugation by μ_x in place of ν), U_y is abelian, and then U is abelian. Since we are assuming that U is not abelian, $\mu_x^2 \neq 1$. By [DS1, Proposition 4.10(5)], $\mu_x^4 = 1$, so (1) holds.

(2): Since $C_U(h)$ is a root subgroup, for all $h \in H$ (see [DST, Corollary 1.9(1)]), it follows in particular that $T_x = C_U(\mu_x^2)$ is a root subgroup. Now the restriction of μ_x to $T_x \cup \{\infty\}$ is the μ -map corresponding to x in $\mathbb{M}(T_x, \mu_x)$ and since the square of this restriction is trivial, part (1) applied to $\mathbb{M}(T_x, \mu_x)$ shows that T_x is abelian. Then, since $C_U(x)$ is a group of exponent p , part (2) holds. $\square$

Lemma 3.3. *Let $x, y \in U^*$ with $|x| = p = |y|$. Then*

- (1) $\eta_x^2 = \mu_x^2$;
- (2) $z\eta_x = -z$, for all $z \in T_x$;
- (3) $\eta_x \in \text{Aut}(U)$ is a regular automorphism of order 4;
- (4) if $y \in T_x$, then $T_x = T_y$. In particular either $T_x = T_y$ or $T_x \cap T_y = 0$.

Proof. (1): We have

$$\begin{aligned} \eta_x^2 &= (\mu_x \mu_{x \cdot \sqrt{-1}})(\mu_x \mu_{x \cdot \sqrt{-1}}) = \mu_x^2 (\mu_{-x \mu_{x \cdot \sqrt{-1}}} \mu_{x \cdot \sqrt{-1}}) \\ &= \mu_x^2 \mu_{-(x \cdot \sqrt{-1}) \mu_x \mu_{x \cdot \sqrt{-1}}} = \mu_x^2 \mu_{-x \cdot \sqrt{-1} \mu_{x \cdot \sqrt{-1}}} = \mu_x^2, \end{aligned}$$

where we have used again the fact that $(x \cdot \sqrt{-1})\mu_x = x \cdot \sqrt{-1}$.

(2): Note first that (e.g. by (1)) η_x acts on T_x . By Lemma 3.2(2), T_x is an abelian root subgroup of U , so by [DS1, Proposition 4.6(6)], $z\mu_{x \cdot \sqrt{-1}} = -z\mu_x$, for all $z \in T_x$. Let $z \in T_x$, then $z\eta_x = z\mu_x \mu_{x \cdot \sqrt{-1}} = -(z\mu_x^2) = -z$.

(3): Suppose that $z\eta_x = z$, for some $z \in U^*$. Then $z\mu_x^2 = z\eta_x^2 = z$. But then $z \in T_x$, so by (2), $z\eta_x = -z$, a contradiction.

(4): Suppose that $y \in T_x$. Since T_x is an abelian root subgroup of U , [DS1, Lemma 5.1] implies that the restriction of μ_y^2 to T_x is trivial, that is $T_x \leq T_y$. By symmetry, $T_y \leq T_x$, so $T_x = T_y$. $\square$

Notice that by Lemma 3.3(1), using the notation of §2, we have

$$T = T_{\eta_a^2} \quad \text{and} \quad S = S_{\eta_a^2}.$$

We can now conclude that

Lemma 3.4. (1) $\eta \in \text{Aut}(U)$ is a regular automorphism of order 4;

- (2) each element $x \in U^*$ can be written uniquely in the form $x = t + s$, with $t \in T$ and $s \in S$;
- (3) $U = \langle S \rangle$.

Proof. Part (1) follows from Lemma 3.3(3). Part (2) follows from Lemma 2.1(2). By [SW], U is characteristically simple, so in particular $[U, U] = U$. Hence part (3) follows from Lemma 2.2(2). $\square$

Proposition 3.5. *Let $s \in S$. Suppose that $|s| \neq p$, then*

- (1) $C_U(s)$ is abelian and $C_U(s)$ is inverted by μ_a^2 .
- (2) $C_U(s) = C_U(x)$, for all $x \in C_U(s)^*$.
- (3) T normalizes $C_U(s)$.

Proof. (1): Since $s\mu_a^2 = -s$, it follows that μ_a^2 acts on $C_U(s)$. Notice that $C_U(s)$ is a uniquely 2-divisible group. We show that $C_{C_U(s)}(\mu_a^2) = 0$, then by Lemma 2.1(3) part (1) will hold. Let $0 \neq x \in C_{C_U(s)}(\mu_a^2)$, then $x \in T$, so $|x| = p$ and $x \in C_U(s)$, so $|x| = |s|$, a contradiction.

(2): Let $x \in C_U(s)$. Then by (1), $x \in S$ and since $|x| = |s|$ we can apply (1) with x in place of s to get that $C_U(x)$ is abelian. Hence (2) follows.

(3): Set $W := C_U(s)$, and let $x \in U^*$. Notice that if $W \cap W^x \neq 0$, then $W = W^x$, because by (2), for $0 \neq y \in W \cap W^x$ we have $W = C_U(y) = W^x$.

Let $t \in T$. By Proposition 2.3(3), $s\eta$ commutes with s , so $s\eta \in W$. By 2.3(4), $[s^t, s\eta] = 0$, so by (2) $s^t \in W$. It follows that $W \cap W^t \neq 0$, and hence $W^t = W$. $\square$

Lemma 3.6. *Let $s \in S$ such that $|s| = p$. Then*

- (1) μ_a^2 commutes with μ_s^2 .
- (2) μ_a^2 inverts T_s and μ_s^2 inverts T .
- (3) η centralizes μ_s^2 , so $T_s\eta = T_s$.
- (4) if $T_s = \langle s \rangle$, then $s\eta = s \cdot i$, for some $i \in \{\sqrt{-1}, -\sqrt{-1}\}$.

Proof. (1): We have $(\mu_s^2)^{\mu_a^2} = \mu_{s\mu_a^2}^2 = \mu_{-s}^2 = \mu_s^2$.

(2): By (1) μ_a^2 acts on T_s . Now $T_s \neq T$, because $s \in T_s \setminus T$. Hence, by Lemma 3.3(4), $T \cap T_s = 0$. Thus μ_a^2 is a regular automorphism of T_s of order 2, so (2) holds. Further, by (1), μ_s^2 acts on T so the same argument shows that μ_s^2 inverts T .

(3): We have

$$\begin{aligned} \eta^{\mu_s^2} &= (\mu_a \mu_{a \cdot \sqrt{-1}})^{\mu_s^2} = \mu_a \mu_s^2 \mu_{(a \cdot \sqrt{-1}) \mu_s^2} \\ &\stackrel{(2)}{=} \mu_{-a} \mu_{-a \cdot \sqrt{-1}} \stackrel{(*)}{=} \mu_a \mu_{a \cdot \sqrt{-1}} = \eta, \end{aligned}$$

where $(*)$ holds since $\mu_{-a}^2 = \mu_{a \cdot \sqrt{-1}}^2$.

(4): By (3) η acts on T_s and since η has no fixed points on T_s , $s\eta \neq s$. Also $s\eta \neq -s$, because $s\eta^2 = s\mu_a^2 = -s$. Since η has order 4, part (4) holds. $\square$

- Proposition 3.7.** (1) Let $s \in S$ and assume that $|s| = p$ and that $T_a = \langle a \rangle$ and $T_s = \langle s \rangle$. Then $[a, s]$ commutes with $\langle a, s \rangle$.
- (2) Assume that for all $x \in U^*$ of order p , $T_x = \langle x \rangle$. Then for all $y \in U^*$, either $y \in S$, or $|y| = p$.
- (3) Assume that for all $x \in U^*$ with $|x| = p$ we have $T_x = \langle x \rangle$. Then U is a p -group.
- (4) Assume that for all $x \in U^*$ with $|x| = p$ we have $T_x = \langle x \rangle$. Then U is abelian.

Proof. (1): By Lemma 3.6(4), $s\eta = s \cdot i$, for some $i \in \{\sqrt{-1}, -\sqrt{-1}\}$, and by Proposition 2.3(4), $s\eta$ commutes with s^a and we see that $[s^a, s] = 0$. Hence $[s, a]$ commutes with s . By symmetry, using Lemma 3.6(2), $[s, a]$ commutes with a .

(2): Let $y \in U^*$ and suppose that $y \notin S$. Using Lemma 3.4(2) and the fact that $T = \langle a \rangle$, we have $y = a \cdot m + s$, with $s \in S$. Assume first that $|s| = p$. Then, by (1), $[a, s]$ commutes with y . Now if $[a, s] = 0$, then clearly $|y| = |a|$, while otherwise $|a| = |[a, s]| = |y|$. In either case $|y| = p$.

So assume that $|s| \neq p$. Then, by Proposition 3.5(3), a normalizes $C_U(s)$, so in particular $y \cdot p \in C_U(s)$. But by Proposition 3.5(1), μ_a^2 inverts $C_U(s)$, so μ_a^2 inverts $y \cdot p$. But if $y \cdot p \neq 0$, this implies that $y \in S$. Since we are assuming that $y \notin S$, we conclude that $y \cdot p = 0$.

(3): Assume there exists $y \in U^*$ with $|y| \neq p$. Then, by (2), $y \in S$. Let $s \in S$. Since $|y^s| = y$ we can apply (2) again to conclude that $y^s \in S$, so

$$s - y - s = (-s + y + s)\mu_a^2 = -(-s + y + s) = -s - y + s.$$

It follows that $s \cdot 2$ and hence s commutes with y . Thus $S \leq C_U(y)$. But by Lemma 3.4(3), $U = \langle S \rangle$, so $y \in Z(U)$. Since U is characteristically simple we get that U is abelian, a contradiction.

(4): By (3), U is a p -group. Let $v \in S$. Then, by Lemma 3.6(4), $v\eta = v \cdot i$, for some $i \in \{\sqrt{-1}, -\sqrt{-1}\}$. Also for each $s \in S$ we have $s\eta \in \{s \cdot i, -s \cdot i\}$. Now $v^a \in S$, so $(v^a)\eta \in \{(v^a) \cdot i, -(v^a) \cdot i\}$.

Suppose that $(v^a)\eta = (v^a) \cdot i = (v \cdot i)^a$. Then,

$$(v \cdot i)^a = (v^a)\eta = (v\eta)^{a\eta} = (v \cdot i)^{-a},$$

and hence $[a, v] = 0$.

Suppose $(v^a)\eta = -(v^a) \cdot i = (-v \cdot i)^a$. Then

$$(-v \cdot i)^a = (v^a)\eta = (v\eta)^{a\eta} = (v \cdot i)^{-a},$$

so $a \cdot 2$ inverts v which is impossible since $|a \cdot 2| = p$ is odd.

Hence a centralizes each element $v \in S$ so $a \in Z(U)$ and since U is characteristically simple, U is abelian. $\square$

As a corollary we get

Proposition 3.8. *There exists $x \in U^*$ such that $|x| = p$ and $T_x \neq \langle x \rangle$. Hence we may (and we will) assume that $T \neq \langle a \rangle$.*

Proof. This follows since we are assuming that U is not abelian and by Proposition 3.7(4). $\square$

We now need the following well known result:

Lemma 3.9. *Let q be a prime and let E be an elementary abelian group of order q^2 . Suppose that E acts on a vector space W . Then there exists $e \in E^*$ such that $C_W(e) \neq 0$.*

Proof. This is well known and follows from [G, Theorem 2.3, p. 65]. $\square$

Proposition 3.10. *Let $s \in S$, then $|s| = p$.*

Proof. Let $s \in S$ and assume that $|s| \neq p$. By Proposition 3.5(3), T normalizes $C_U(s)$. Let $E \leq T$ be a subgroup of order p^2 . The existence of E follows from Proposition 3.8 and Lemma 3.2(2).

Of course E is elementary abelian. Since E acts on $C_U(s)$, Lemma 3.9 together with Proposition 3.5(1), implies that there exists $w \in C_U(s)^*$ and $e \in E^*$ such that $[w, e] = 0$. But then $|s| = |w| = |e| = |a|$, a contradiction. $\square$

Lemma 3.11. *Let $s \in S$, so that by Proposition 3.10 $|s| = p$. Assume that $s\eta = s \cdot i$, for some $i \in \{\sqrt{-1}, -\sqrt{-1}\}$. Then*

- (1) $T_{s^a} = T_{s^{-a}}$.
- (2) $(T_{s^a})\mu_a^2 = T_{s^a} = (T_{s^a})\mu_s^2$.
- (3) $[s, a] = 0$.

Proof. (1): Notice first that

$$(3.2) \quad \text{for each } h \in H \text{ and } x \in U^* \text{ we have } T_{xh} = (T_x)h.$$

Indeed

$$v \in T_{xh} \iff v\mu_{xh}^2 = v \iff vh^{-1}\mu_x^2h = v \iff vh^{-1} \in T_x.$$

Since $s^a \in S$, Lemma 3.6(3) implies that $(T_{s^a})\eta = T_{s^a}$. On the other hand, by (3.2),

$$(T_{s^a})\eta = T_{(s^a)\eta} = T_{(s \cdot i)^{-a}} = T_{(s)^{-a} \cdot i} = T_{s^{-a}},$$

where the last equality follows from the fact that $\mu_{(s^{-a}) \cdot i}^2 = \mu_{s^{-a}}^2$.

(2): By (3.2), we have $(T_{s^a})\mu_a^2 = T_{(s^a)\mu_a^2} = T_{(-s)^a} = T_{s^a}$. Also, $(T_{s^a})\mu_s^2 = T_{(s^a)\mu_s^2} = T_{s^{-a}} = T_{s^a}$, by (1), where we have used the fact that by Lemma 3.6(2), $a\mu_s^2 = -a$.

(3): Notice that by Lemma 3.6(1), $E := \langle \mu_a^2, \mu_s^2 \rangle$ is an elementary abelian group of order 4 and that $E^* = \{\mu_a^2, \mu_s^2, \mu_{a\mu_s}^2 = \mu_a^2\mu_s^2\}$. Since E acts on T_{s^a} , which is a (possibly infinite) elementary abelian p -group, Lemma 3.9 implies

that there exists $e \in E^*$ such that $C_{T_{s^a}}(e) \neq 0$. It follows that there exists $x \in \{a, s, a\mu_s\}$ such that $T_{s^a} \cap T_x \neq 0$, so by Lemma 3.3(4) $T_{s^a} = T_x$.

Suppose $x = a$, then $s^a = s^a \mu_a^2 = (-s)^a$ which is false. Suppose $x = a\mu_s$. Then $s^a = (s^a) \mu_a^2 \mu_s^2 = ((-s)^a) \mu_s^2 = (-s)^{-a}$. This is also impossible. Hence $s^a = (s^a) \mu_s^2 = s^{-a}$ and we see that $[a, s] = 0$. $\square$

We can now prove Theorem A.

Proof of Theorem A. We assume that U is not abelian. Let $s \in S$. By Proposition 3.10, $|s| = p$. By Lemma 3.6(3), $T_s \eta = T_s$. For $i \in \{\sqrt{-1}, -\sqrt{-1}\}$, let

$$T_{s,i} := \{v \in T_s \mid v\eta = v \cdot i\}$$

We claim that

$$T_s = T_{s,\sqrt{-1}} \oplus T_{s,-\sqrt{-1}}.$$

Notice that $\eta \in \text{Aut}(T_s)$ is an operator of order 4 and that 1 and -1 are not eigenvalues of η . The fact that 1 is not an eigenvalue follows since η is regular on U and the fact that -1 is not an eigenvalue of η follows since $\eta^2 = \mu_a^2$ inverts T_s (see Lemma 3.6(2)). Hence i and $-i$ are the only eigenvalues of η and the claim holds.

Notice that by Lemma 3.6(2), $T_s \subseteq S$. Let now $v \in T_{s,i}$. Then, by Lemma 3.11(3), $[v, a] = 0$. It follows that $[a, T_s] = 0$ and in particular $[a, s] = 0$. Since $s \in S$ was arbitrary we see that $[a, S] = 0$. But by Lemma 3.4(3), $U = \langle S \rangle$, so $a \in Z(U)$ and since U is characteristically simple, U is abelian, a contradiction. $\square$

Acknowledgment. It is my pleasure to thank Tom De Medts for a careful reading of the paper and for pointing out to me a number of typos.

REFERENCES

- [DS1] T. De Medts, Y. Segev, *Identities in Moufang sets*, Trans. Amer. Math. Soc, to appear.
- [DS2] T. De Medts, Y. Segev, *A course on Moufang sets*, Innov. Incidence Geom., to appear.
- [DST] T. De Medts, Y. Segev, K. Tent, *Special Moufang sets, their root groups and their μ -maps*, Proc. Lond. Math. Soc. **96** (2008), no. 3, 767–791.
- [DW] T. De Medts and R. M. Weiss, *Moufang sets and Jordan division algebras*, Math. Ann. **335** (2006), no. 2, 415–432.
- [G] D. Gorenstein, *Finite Groups*, Chelsea Publishing Co., New York, 1980.
- [K] L. G. Kovács, *Groups with regular automorphisms of order 4*, Math. Zeitschr. **75** (1961), 277–294.
- [S1] Y. Segev, *Finite special Moufang sets of odd characteristic*, Commun. Contemp. Math., **10** (2008), 455–475.
- [S2] Y. Segev, *Proper Moufang sets with abelian root groups are special*, submitted.
- [SW] Y. Segev, R. M. Weiss, *On the action of the Hua subgroups in special Moufang sets*, Math. Proc. Cambridge Philos. Soc. **144** (2008), no. 1, 77–84.
- [Ti] F. Timmesfeld, *Abstract root subgroups and simple groups of Lie type*, Birkhäuser-Verlag, *Monographs in Mathematics* **95** Basel, Berlin, Boston, 2001.

- [T] J. Tits, *Twin buildings and groups of Kac-Moody type*, in *Groups, combinatorics & geometry (Durham, 1990)*, 249–286, London Math. Soc. Lecture Note Ser. **165**, Cambridge Univ. Press, Cambridge, 1992.

YOAV SEGEV, DEPARTMENT OF MATHEMATICS, BEN-GURION UNIVERSITY, BEER-SHEVA 84105, ISRAEL

E-mail address: `yoavs@math.bgu.ac.il`