ON THE ACTION OF THE HUA SUBGROUPS IN SPECIAL MOUFANG SETS

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Abstract. We show that in a special Moufang set, either the root groups are elementary abelian 2-groups, or the Hua subgroup $H$ (= the Cartan subgroup) acts “irreducibly” on $U$, i.e. the only non-trivial $H$-invariant subgroup of a root group normalized by $H$ is the whole root group.

1. Introduction

A Moufang set is a permutation group $G^\dagger$ on a set $X$ together with a conjugacy class of subgroups $\{U_x \mid x \in X\}$ that generate $G^\dagger$ such that $U_x$ fixes $x$, acts regularly on $X \setminus \{x\}$ and $U_x^\varphi = U_{x\varphi}$, for all $\varphi \in G^\dagger$. Note that $G^\dagger$ is a doubly transitive permutation group. The subgroups $\{U_x \mid x \in X\}$ are called the root groups of the Moufang set. This name comes from the fact that almost all known Moufang sets are essentially the buildings associated to simple algebraic groups of relative rank one, where the groups $U_x$ are, in fact, root groups.

Any Moufang set can be constructed as follows (see [DW]). Start with a group $U$ and let $\infty$ be a new symbol (not in $U$). Let $X$ denote the set $X := U \cup \{\infty\}$.

We write $U$ in additive notation even though we do not assume that $U$ is commutative. For $a \in U^* := U \setminus \{0\}$ we let $\alpha_a \in \text{Sym}(X)$ be the permutation

$$x\alpha_a := \begin{cases} x + a & \text{if } x \in U; \\ \infty & \text{if } x = \infty. \end{cases}$$

(We compose functions throughout this paper from left to right and write all functions on the right.)

Suppose that $\tau \in \text{Sym}(X)$ with $0\tau = \infty$ and $\infty\tau = 0$, let

$$U_\infty = \{\alpha_a \mid a \in U\}, U_0 = U_\infty^\tau,$$

and $U_a = U_0^{\tau a}$ for all $a \in U^*$

and let $G^\dagger = \langle U_x \mid x \in X \rangle$. We denote the group $G^\dagger$ together with the subgroups $\{U_x \mid x \in U\}$ by $\text{M}(U, \tau)$. For each $a \in U^*$, let

$$\mu_a := \alpha_{(-a)\tau^{-1}}\alpha_a\alpha_{(a\tau^{-1})}^{-1}$$

and

$$h_a = \tau\mu_a.$$
where we write $g^h$ to denote $h^{-1}gh$ for group elements $g, h$. Following [DW] we call $h_a$ the *Hua-maps* (of $\mathcal{M}(U, \tau)$ corresponding to $\tau$). For each $a \in U^*$, $h_a$ fixes $\infty$ and 0 and hence acts as a permutation on the set $U^*$. The main result (Theorem 2) of [DW] says that $\mathcal{M}(U, \tau)$ is a Moufang set if and only if for all $a \in U^*$, $h_a \in \text{Aut}(U)$.

Suppose that $\mathcal{M}(U, \tau)$ is a Moufang set. Then for each $a \in U^*$, the permutation $\mu_a$ is the unique element of $U_0\alpha_aU_0$ that interchanges $\infty$ and 0 (see Lemma 3.3(2) in [DS]).

It follows from this (or directly from the definition) that $\mu_a^{-1} = \mu_{-a}$ for all $a \in U^*$. We set

$$H := G^\dagger_{0,\infty},$$

the pointwise stabilizer in $G^\dagger$ of 0 and $\infty$. For reasons explained in [DW], the subgroup $H$ (or any subgroup of $G^\dagger$ conjugate to $H$) is called a *Hua subgroup*. (In the examples which are the buildings associated to a simple algebraic group of relative rank one, $H$ is, in fact, a Cartan subgroup.) As was shown in Theorem 1(ii) of [DW],

$$H = \langle \mu_a\mu_b \mid a, b \in U^* \rangle.$$

**Definition 1.1.** A Moufang set $\mathcal{M}(U, \tau)$ is called *special* if the condition

$$(-a)\tau = -(a\tau)$$

for all $a \in U^*$ holds.

Throughout this note, we assume that $\mathcal{M}(U, \tau)$ is a special Moufang set. Thus

$$(-x)\mu_y = (-x)(\tau\mu_{-y})^{-1}\tau = -(x(\tau\mu_{-y})^{-1}\tau) = -(x\mu_y)$$

for all $x, y \in U^*$, since $\tau\mu_{-y} \in \text{Aut}(U)$. By Lemma 4.3(2) of [DS],

$$a\mu_a = -a = a\mu_{-a}, \text{ for all } a \in U^*.$$  

(1.1)

It follows that

$$\text{the order of } a\mu_x \text{ is equal to the order of } a \text{ for all } a, x \in U^*,$$

(1.2)

(and if one is infinite then so is the other). The reason is that $a\mu_x = (-a)\mu_a\mu_x$ and $\mu_a\mu_x \in \text{Aut}(U)$.

Our main goal in this note is to show the following:

**Theorem 1.2.** Let $\mathcal{M}(U, \tau)$ be a special Moufang set. Let $W \leq U$ be a nontrivial $H$-invariant subgroup. Then either $U$ is an elementary abelian 2-group, or $W = U$.

See also Corollaries 3.1 and 3.2 and Proposition 3.4. As we will see at the end of §2, the case when $U$ is an elementary abelian 2-group in Theorem 1.2 is a genuine exception: Example 2.7 shows that there are special Moufang sets $\mathcal{M}(U, \tau)$ such that $U$ is an elementary abelian 2-group, but the action of $H$ on $U$ is not irreducible.

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2. HUA-invariant subgroups of $U$

In this section let $W$ be a nontrivial $H$-invariant subgroup of $U$. Notice that since for each $w \in W$ and $u \in U^*$, one has $w \mu_u = (-w)\mu_u$ and $\mu_u \in H$, it follows that

$$W \mu_u = W \text{ for all } u \in U^*.$$

We start by recalling that

**Lemma 2.1** ([DS] Lemma 4.4(3)). Let $\mathcal{M}(U, \tau)$ be a special Moufang set. Then

$$c\mu_{c+d} = -d - c + c\mu_d - d \text{ for all } c, d \in U^* \text{ such that } c + d \neq 0.$$

Recall also that by [DS, Proposition 4.6] the following holds:

**Proposition 2.2.** Assume that $\mathcal{M}(U, \tau)$ is a special Moufang set. Let $a \in U^*$, $n \geq 1$ be a positive integer such that $a \cdot n \neq 0$, and $\rho \in \text{Sym} \langle X \rangle$ be a permutation interchanging $0$ and $\infty$ such that $\mathcal{M}(U, \rho) = \mathcal{M}(U, \tau) = \mathcal{M}(U, \rho^{-1})$ (which holds, in particular, if $\rho = \mu_x$, $x \in U^*$). Then

1. there exists a unique $b \in U^*$, which we will denote by $a \cdot \frac{1}{n}$, such that $b \cdot n = a$;
2. $(ap) \cdot n \neq 0$; $(a \cdot n)\rho = (ap) \cdot \frac{1}{n}$, and hence $(a \cdot \frac{1}{n})\rho = (ap) \cdot n$;
3. if $U$ is torsion free, then $U$ is a uniquely divisible group;
4. if $b \in U^*$ has finite order, then the order of $b$ is a prime number.

**Lemma 2.3.** Let $a \in W^*$ and $b \in U$. If $b + a + b \in W$, then $b \cdot n + a + b \cdot n \in W$ for all $n \in \mathbb{Z}$.

**Proof.** Assume first that for some $k, m \in \mathbb{Z}$, $b \cdot k + a + b \cdot m = 0$. Then $-b \cdot (k + m) = a \in W$, so, by equation (2.1), $(b \cdot (k + m))\mu_b \in W$. It follows that $(b \cdot (k + m))\mu_b \cdot (k + m) \in W$. Also, by Proposition 2.2(2), $(c \cdot n)\rho \cdot n = c\rho$, for $c \in U^*$ with $c \cdot n \neq 0$. Applying this to $c = b$, $n = k + m$ and $\rho = \mu_b$ we get that $b\mu_b = (b \cdot (k + m))\mu_b \cdot (k + m) \in W$. By equation (1.1), $b\mu_b = -b$ and it follows that $b \in W$, so the lemma clearly holds. Thus we may assume that $b \cdot k + a + b \cdot m \neq 0$ for all $k, m \in \mathbb{Z}$. Of course we may assume that $b \neq 0$.

Next, applying Lemma 2.1 to $c = b + a + b$ and $d = -(a + b)$, and recalling from the introduction that $\mu_{-a} = \mu_{a+b}^{-1}$, we conclude that

$$(b + a + b)\mu_{(b+a+b)+(-b-a)} = a + b - (b + a + b) + (b + a + b)\mu_{a+b}^{-1} + a + b$$

$$= -b + (b + a + b)\mu_{a+b}^{-1} + a + b.$$

Then, as $b = (b + a + b) + (-b - a)$, we have

$$b + a + b)\mu_b = -b + (b + a + b)\mu_{a+b}^{-1} + a + b.$$

(*)
By (\textit{*)},
\begin{align*}
a &= -(b + a + b)\mu_{a+b}^{-1} + b + (b + a + b)\mu_b - b \\
&= -(b + a + b)\mu_{a+b}^{-1} + b + (b + a + b) + b - b - (b + a + b) + (b + a + b)\mu_b - b \\
&= -(b + a + b)\mu_{a+b}^{-1} + [b + (b + a + b) + b] + [-b - (b + a + b) + (b + a + b)\mu_b - b] \\
&= -(b + a + b)\mu_{a+b}^{-1} + [b \cdot 2 + a + b \cdot 2] + [-b - (b + a + b) + (b + a + b)\mu_b - b].
\end{align*}

Now, using Lemma 2.1 with \(c = b + a + b\) and \(d = b\), we get that
\[-b - (b + a + b) + (b + a + b)\mu_b - b = (b + a + b)\mu_{b+a+b},\]
so we may continue our equalities to get
\[a = -(b + a + b)\mu_{a+b}^{-1} + [b \cdot 2 + a + b \cdot 2] + [(b + a + b)\mu_{b+a+b}].\]

By equation (2.1), therefore, \(b \cdot 2 + a + b \cdot 2 \in W\). By induction on \(n\) we have: \(b \cdot n + a + b \cdot n \in W\) for all \(0 \leq n \in \mathbb{Z}\). Further, if we set \(x := b + a + b\), then we see that \(x\) and \(-b + x + (-b) \in W\), so by the above \((-b) \cdot n + x + (-b) \cdot n \in W\) for all \(0 \leq n \in \mathbb{Z}\), and hence, \(b \cdot n + a + b \cdot n \in W\) for all \(n \in \mathbb{Z}\).

**Corollary 2.4.** Let \(a \in W^*\) and \(b \in U^*\). If \(a + b \neq 0\), then \(b \cdot n + a\mu_{a+b} + b \cdot n \in W\) for all \(n \in \mathbb{Z}\).

**Proof.** As \(a \in W\), equation (2.1) says that \(a\mu_b\) and \(a\mu_{a+b}\) are also in \(W\), so \(-a + a\mu_b \in W\). Then, by Lemma 2.1, \(b + a\mu_{a+b} + b = -a + a\mu_b \in W\), so the corollary follows from Lemma 2.3.

**Proposition 2.5.** Assume \(W\) is normal in \(U\). Then either \(U\) is an elementary abelian 2-group or \(W = U\).

**Proof.** We repeatedly use the fact (that we saw in equation (2.1)) that for \(x \in U^*, W\mu_x = W\) (and hence if \(u \in U \setminus W\), then \(u\mu_x \not\in W\)). We assume that \(W \neq U\) and we show that \(U\) is an elementary abelian 2-group.

First we show that
\[(i) \quad \text{if } w \in W \text{ and } w \cdot 2 \neq 0, \text{ then } w \cdot \frac{1}{2} \in W.\]

Let \(w \in W\) such that \(w \cdot 2 \neq 0\). Then, by Proposition 2.2(2), with \(\rho = \mu_w\) and using equation (1.1), we have \(w \cdot \frac{1}{2} = ((-w) \cdot 2)\mu_w\). Hence \(w \cdot \frac{1}{2} \in W\).

Next we claim that
\[(ii) \quad \text{if } u \in U \text{ and } u \cdot 2 \neq 0, \text{ then } u \in W.\]

Let \(u \in U\) with \(u \cdot 2 \neq 0\) and choose \(w \in W^*\) such that \(u + w \neq 0\). By Lemma 2.1, \(w\mu_{w+u} = -u - w + w\mu_u - u\). Notice however that \(w, w\mu_{w+u}, w\mu_u \in W\), and since \(W\) is normal in \(U\) it follows that \(w\mu_{w+u}\) conjugated by \(u\) is in \(W\). Hence \(-u \cdot 2 = (-u + w\mu_{w+u} + u) - (-w + w\mu_u) \in W\). But \(u \cdot 4 \neq 0\), since there are no elements of order 4 in \(U\) (see Proposition 2.2(4)). It follows from (i) that \(u = (u \cdot 2) \cdot \frac{1}{2} \in W\).
Our next step is to show that

(iii) \( u \in U \setminus W \), then \( u \) inverts \( W \); in particular \( W \) is abelian.

By (ii) we see that all elements in \( U \setminus W \) are involutions. It follows that any involution \( u \in U \setminus W \) inverts \( W \), because \( w + u \notin W \) for \( w \in W \), and then \( w + u \) is an involution, so \( u \) inverts \( w \) (i.e. \( u + w + u = -w \)). This implies that \( W \) is abelian.

Next we claim

(iv) \( W \) is an elementary abelian 2-group, and hence so is \( U \).

If \( W \) is an elementary abelian 2-group, then, since by (ii), all elements in \( U \setminus W \) are involutions, we see that \( U \) is also an elementary abelian 2-group and we are done.

So assume that \( W \) is not an elementary abelian 2-group. Let \( x, y \in U \setminus W \). Since \( x \) and \( y \) invert \( W \), \( x + y \) centralizes \( W \). But if \( x + y \notin W \), then \( x + y \) inverts \( W \). It follows that \( x + y \in W \) and thus \( W \) has index 2 in \( U \). Let now \( x, y \in U \setminus W \) be elements such that \( x + y \neq 0 \). Then, by Lemma 2.1, \(-x\mu_{x+y} - y - x + x\mu_y - y = 0 \). However, \( x, y, x\mu_y, y\mu_y \notin W \), so we get that 0 is the sum of an odd number of elements which are not in \( W \). This contradicts the fact that \( U/W \) has order two. Hence (iv) holds and the proof of the proposition is complete. \( \square \)

Our next result is

**Lemma 2.6.** If \( W \neq U \), then \( W \) is an elementary abelian 2-group.

*Proof.* Let \( a \in W^* \) and \( b \in U \setminus W \). By Lemma 2.1 and equation (2.1) we have

\[
W \ni a\mu_{a+b} = -(a + b) - a + a\mu_{a+b} - (a + b) = -b - a \cdot 2 + a\mu_{a+b} - b - a,
\]

and it follows that

\[
-b - a \cdot 2 + b - b + a\mu_{a+b} - b \in W \text{ for all } a \in W^* \text{ and } b \in U \setminus W.
\]

But by Corollary 2.4, \(-b + a\mu_{a+b} - b \in W \) and it follows that \(-b + a \cdot 2 + b \in W \). Suppose now that \( a \cdot 2 \neq 0 \), and set \( u := -b + a + b \). Then \( u \cdot 2 = -b + a \cdot 2 + b \in W \).

By Proposition 2.2(2), equation (1.1) and equation (2.1), \( u = ((-u) \cdot 2)\mu_a \cdot 2 \in W \).

We have shown that for \( a \in W^* \), each conjugate of \( a \cdot 2 \) under \( U \) is a square in \( W \), so the subgroup

\[
V := \langle w \cdot 2 \mid w \in W \rangle,
\]

is a normal subgroup of \( U \). Note that \( V \) is also \( H \)-invariant because \( V \) is a characteristic subgroup of \( W \). Hence, by Proposition 2.5, either \( V = U \), or \( V = 0 \). It follows that either \( W = U \) or \( W \) is an elementary abelian 2-group. \( \square \)

**Proof of Theorem 1.2:** Suppose \( W \neq U \). By Lemma 2.6 all nontrivial elements of \( W \) are involutions. We show in a number of steps that then every element in \( U \setminus W \) is an involution, it will then follow that \( U \) is an elementary abelian 2-group.

So let \( b \in U \setminus W \) and assume that \( b \cdot 2 \neq 0 \). We will eventually show that such a \( b \) does not exist by obtaining a contradiction.

(i) \( a\mu_{a+b} \) inverts \( b \), and hence \( a\mu_{a+b} = a + a\mu_b \) for all \( a \in W^* \).
By Corollary 2.4, \( b + a\mu_{a+b} + b \in W \), so \( b + a\mu_{a+b} + b \) is an involution. This implies that \( a\mu_{a+b} \) inverts \( b \cdot 2 \). However, by Proposition 2.2(1), \(-b\) is the unique square root of \(-b \cdot 2\) in \( U \), so \( a\mu_{a+b} \) inverts \( b \). As \( a\mu_{a+b} \) inverts \( b \) we have \( b + a\mu_{a+b} + b = a\mu_{a+b} \), so the second part of (i) holds by Lemma 2.1.

Next we show

\[(ii) \quad \text{if } a \in W^* \text{ inverts } b, \text{ then } a\mu_b \text{ centralizes } b.\]

By (i), \( a\mu_b = a + a\mu_{a+b} \) and \( a\mu_{a+b} \) inverts \( b \). So if \( a \) inverts \( b \), then \( a\mu_b \) centralizes \( b \).

\[(iii) \quad b\mu_{b+a} \text{ centralizes } a \text{ and hence } b\mu_{b+a} = -b + b\mu_a \text{ for all } a \in W^*.\]

By (i) \( a\mu_{a+b} \) inverts \( b \), so by (ii), \( a\mu_{a+b}\mu_b \) centralizes \( b \). Recalling that \( \mu_{a+b}\mu_b \in \text{Aut}(U) \) we see that \( b\mu_{b}^{-1}\mu_{a+b}^{-1} \) centralizes \( a \), that is \( -(b)\mu_{-a-b} \) centralizes \( a \) (because \( \mu_a^{-1} = \mu_{-a} \) and using equation (1.1)). Replacing \( b \) with \(-b \) we get the first part of (iii). Now \( b\mu_{b+a} = a + b\mu_{b+a} + a = -b + b\mu_a \) by Lemma 2.1, and using the fact that \( a + a = 0 \) (Lemma 2.6).

\[(iv) \quad \text{If } b \text{ centralizes } a \in W^*, \text{ then } b\mu_a \text{ centralizes } a.\]

By (iii) we have \( b\mu_a = b + b\mu_{b+a} \), so if \( b \) centralizes \( a \) then by (iii) \( b\mu_a \) centralizes \( a \).

We can now obtain our desired contradiction. Choose \( a \in W^* \). By (iii) \( b\mu_{b+a} \) centralizes \( a \). By equation (1.2) the order of \( b\mu_{b+a} \) is distinct from 2, so we can apply (iv) with \( b\mu_{b+a} \) in place of \( b \). Thus, by (iv), \( b\mu_{b+a}\mu_a \) centralizes \( a \). Since \( \mu_{b+a}\mu_a \in \text{Aut}(U) \), we get that \( a\mu_a^{-1}\mu_{-(b+a)} \) centralizes \( b \), i.e. \( a\mu_{a-b} \) centralizes \( b \). But by (i), \( a\mu_{a-b} \) inverts \( b \), so \( b \) must be an involution. This contradicts our hypothesis that \( b \) is not an involution and completes the proof of Theorem 1.2. \( \square \)

We now give an example of a special Moufang set \( \mathcal{M}(U, \tau) \) such that \( U \) is an elementary abelian 2-group, and such that the action of \( H \) on \( U \) is not irreducible.

**Example 2.7.** Let \( q \) be an anisotropic quadratic form on a vector space \( U \) over a field \( k \), let \( f \) denote the corresponding bilinear form and let \( \tau \) be the map from \( U^* \) to itself given by \( x\tau = x/q(x) \). Let also \( 0\tau = \infty \) and \( \infty\tau = 0 \). Then the Hua-maps of \( \mathcal{M}(U, \tau) \) corresponding to \( \tau \) are given by

\[ xh_a = x\pi_a \cdot q(a) \]

for all \( x \in U \) and all \( a \in U^* \), where

\[ x\pi_a = x - f(x, a)q(a). \]

In particular, \( h_a \) is additive for each \( a \in U^* \). Hence by Theorem 2 of [DW], \( \mathcal{M}(U, \tau) \) is a Moufang set. By Lemma 8(ii) of [DW], we have

\[ x\mu_a = x\pi_a \cdot q(a)/q(x) \]

for all \( x, a \in U^* \). Thus

\[ x\mu_a\mu_b = x\pi_a\pi_b \cdot q(b)/q(a) \]
for all $x, a, b \in U^*$. Now suppose that the radical

$$R := \{ u \in U \mid f(u, v) = 0 \text{ for all } v \in U \}$$

of $f$ is non-zero. This can happen only if char $k = 2$ since otherwise $f(u, u) = q(u)/2 \neq 0$ for all $u \in U^*$. Since the maps $\pi_a$ preserve $q$ and therefore $f$, the maps $\mu_a \mu_b$ normalize $R$. Thus (assuming also $R \neq U$) $R$ is a non-trivial $H$-invariant subgroup of $U$. Explicit examples of such quadratic forms are described, for example, in [TW], (14.23).

3. Applications

In this section we give three consequences of Theorem 1.2. By [Ti, Thm. 5.2(a), p. 55] if $U$ is abelian, then $U$ is a vector space over $\mathbb{Q}$ or over $\text{GF}(p)$, for some prime $p$. Let

$$\mathfrak{H} \subseteq \text{End}(U),$$

be the centralizer in $\text{End}(U)$ of $H$, when $U$ is abelian. We register the following immediate corollary to Theorem 1.2. It says that when $U$ is abelian but not of exponent 2, the center of $\mathfrak{H}$ is a “natural” base field for the vector space $U$.

**Corollary 3.1.** Let $M(U, \tau)$ be a special Moufang set and assume that $U$ is abelian but not of exponent 2. Then $\mathfrak{H}$ is a skew-field and hence $U$ is a vector space over $\mathfrak{H}$.

**Proof.** By Theorem 1.2 $H$ is irreducible on $U$, so the result follows from Schur’s lemma. $\square$

It is conjectured that every Moufang set has nilpotent root groups. Modulo this conjecture, the following corollary shows that every special Moufang set has abelian root groups.

**Corollary 3.2.** Let $M(U, \tau)$ be a special Moufang set and assume that $U$ is nilpotent. Then $U$ is abelian.

**Proof.** Since the center of $U$ is a non-trivial characteristic subgroup of $U$, it is $H$-invariant and hence, by Theorem 1.2, must equal $U$. $\square$

We now consider the case that $U$ is finite. To deal with this case, we first need the following lemma.

**Lemma 3.3.** Suppose $M(U, \tau)$ is a special Moufang set. Then

1. if $a, b$ are involutions in $U$, then $a$ commutes with $a\mu_b$;
2. $U$ is not isomorphic to the alternating group $A_5$.

**Proof.** (1): By equation (1.2), $a\mu_b$ is an involution. By Lemma 2.1,

$$a\mu_{a+b} = b + a + a\mu_b + b,$$

and since $a\mu_{a+b}$ is an involution we see that

$$b + a + a\mu_b + b = a\mu_{a+b} = -a\mu_{a+b} = b + a\mu_b + a + b,$$
so (1) holds.

(2): Assume $U \cong A_5$, let $a, b \in U^*$ be distinct involutions and consider $\varphi := \mu_a\mu_b$. We know that $\varphi \in \text{Aut}(U)$. We claim that by (1),

(i) $x\varphi$ commutes with $x$ for each involution $x \in U^*$.

Indeed, by (1), $x\mu_a$ commutes with $x$ and $x\mu_a\mu_b$ commutes with $x\mu_a$. It follows that $x$ and $x\varphi$ commute with $x\mu_a$ and this implies that $x$ and $x\varphi$ commute, since the centralizer of an involution in $A_5$ is abelian. But the only automorphism $\varphi \in \text{Aut}(U) \cong S_5$ that satisfies (i) is the identity automorphism. Thus $\mu_a = \mu_b$ (note that $\mu_a$ and $\mu_b$ have order 2 as $\mu_a = \mu_a^{-1}$). By [DS, Proposition 4.9(4)] this implies $a = b$, a contradiction. □

**Proposition 3.4.** Assume $\mathbb{M}(U, \tau)$ is a special Moufang set and that $U$ is finite. Then $U$ is abelian.

*Proof.* Assume that $U$ is not abelian. Consider the generalized Fitting subgroup $F^*(U)$. Since it is a characteristic subgroup of $U$, Theorem 1.2 implies that $U = F^*(U)$. Next, since the Fitting subgroup $F(U)$ of $U$ is a characteristic subgroup of $U$, Theorem 1.2 and Corollary 3.2 imply that $F(U) = 0$.

Hence $U$ is a direct product of simple groups. Since the order of every element of $U$ is a prime number (Proposition 2.2(4)), $U$ must be a non-abelian simple group. By the Odd Order Theorem [FT], the order of $U$ is even. Now the 2-Sylow subgroup of $U$ must be elementary abelian ($U$ has no elements of order 4). Furthermore, if $S \in \text{Syl}_2(U)$, then the centralizer of every nontrivial element $a \in S$ equals $S$. This is because by Proposition 2.2(4), $C_U(a)$ must be an elementary abelian 2-group, and since $S \subseteq C_U(a)$, it follows that $S = C_U(a)$. Thus two distinct 2-Sylow subgroups of $U$ have only the identity in common, and $S$ is elementary abelian. By a theorem of M. Suzuki [Su], $U \cong \text{SL}_2(2^n)$. But $\text{SL}_2(2^n)$ contains cyclic subgroups of order $2^n - 1$ and $2^n + 1$. If $n > 2$, is even then $2^n - 1$ is not a prime and if $n > 2$ is odd, then 3 divides $2^n + 1$ so $2^n + 1$ is not a prime. Thus by Proposition 2.2(4), $n = 2$. However, by Lemma 3.3, $U \not\cong A_5 \cong \text{SL}_2(4)$. This contradiction shows that $U$ must be abelian. □

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